# Computational Control

## Economic MPC

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# Steady-state selection in MPC

$$
\begin{aligned}
\min_{\mathbf{u},\mathbf{x}} \quad & \sum_{k=0}^{T-1} \left( x_t^\top Q x_t + u_t^\top R u_t \right) + x_T^\top S x_T, \qquad Q, S \succeq 0,\ R \succ 0 \\
\text{subject to} \quad & x_{t+1} = A x_t + B u_t \\
& \mathbf{x} \in \mathcal{X} \\
& \mathbf{u} \in \mathcal{U}
\end{aligned}
$$

We have implicitly assumed that we are trying to regulate the steady state

$$x_s = 0 \qquad u_s = 0$$

However

- we usually want plants to operate at a desireable working point $x_s$ (not to stop! $x_s = 0$)
- we can often allow a constant input $u_S$

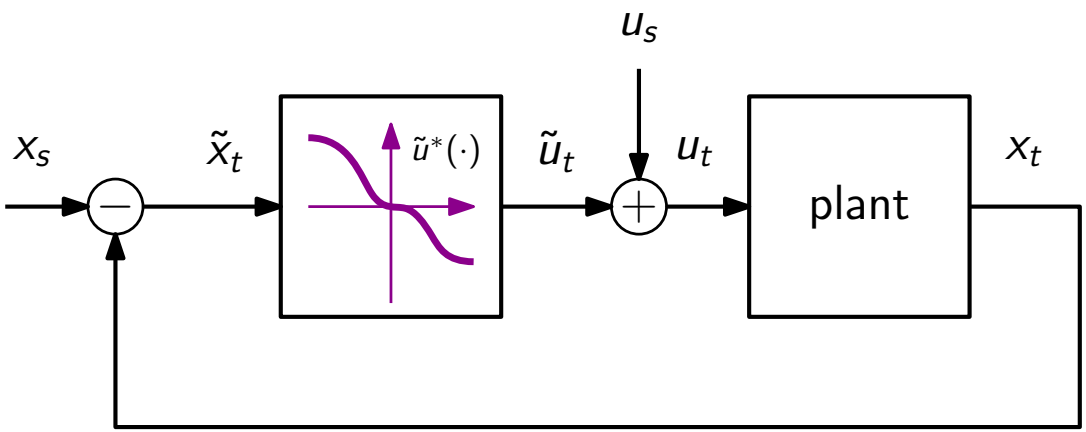
# Non-zero steady state

If a desired equilibrium $(x_s, u_s)$ is known, MPC can be simply be used to stabilize the system around that point.

**change of coordinates** $\begin{cases} \tilde{x} = x - x_s \\ \tilde{u} = u - u_s \end{cases}$

For linear systems, the change of variables is **inconsequential**.

$$V(x_0) = \min_{\tilde{\mathbf{x}}, \tilde{\mathbf{u}}} \sum_{k=0}^{K-1} \left( \tilde{x}_k^\top Q \tilde{x}_k + \tilde{u}_k^\top R \tilde{u}_k \right) + \tilde{x}_K^\top S \tilde{x}_K \qquad \text{subject to} \quad \tilde{x}_{k+1} = A\tilde{x}_k + B\tilde{u}_k$$

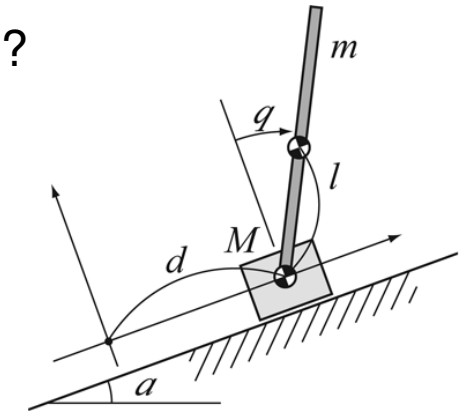


# Steady-state selection

How do you decide the steady state (equilibrium) $(x_s, u_s)$?

In some cases, it can be derived from first principles.

$$q = a, \quad \dot{q} = 0, \quad d = D, \quad \dot{d} = 0, \quad u = g \sin a$$

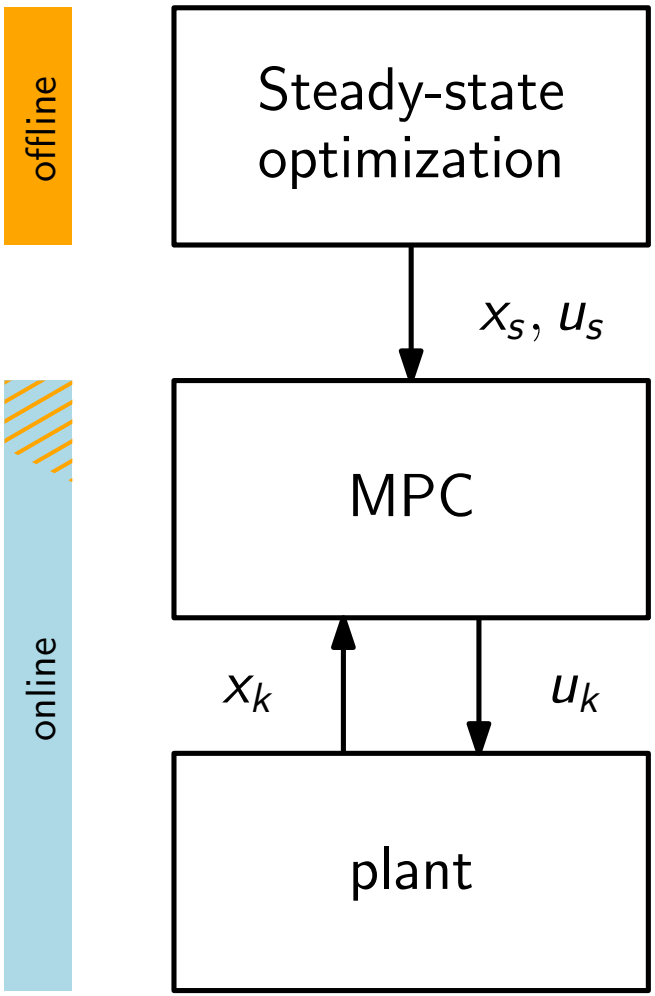



In other cases, specifications indicate a "desired" steady state $(x_{\text{spec}}, u_{\text{spec}})$.

## Steady-state selection problem

$$\begin{aligned} \min_{x_s, u_s} \quad & \|x_s - x_{\text{spec}}\|_{Q_s}^2 + \|u_s - u_{\text{spec}}\|_{R_s}^2 \\ \text{subject to} \quad & \begin{bmatrix} I - A & -B \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = 0 \quad \textbf{(a valid steady state)} \\ & x_s \in \mathcal{X} \\ & u_s \in \mathcal{U} \end{aligned}$$

# Tracking problems




For example

- vehicle guidance
- robotic motion guidance
- low-level objectives in industrial processes, such as regulating
  - pressures
  - temperatures
  - levels
  - flows
- electro-mechanical drives

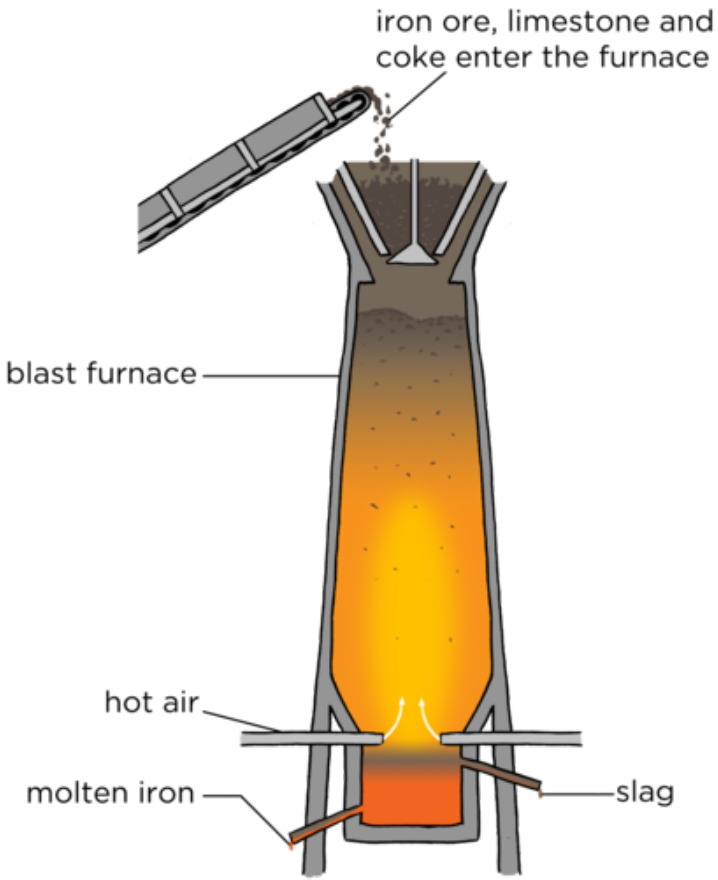
Especially in the presence of

- known dynamics
- constraints
- high-dimensional systems

# Higher-level goals

Many modern control problems present **complex performance metrics**, such as

- economic aspects
- regulatory specifications
- use of resources




Inputs

- furnace temperature
- inflow of iron ore / limestone / coke
- air flow

Output

- iron yield
- iron quality
- $CO_2$ emissions
- amount of slag

# Economic MPC

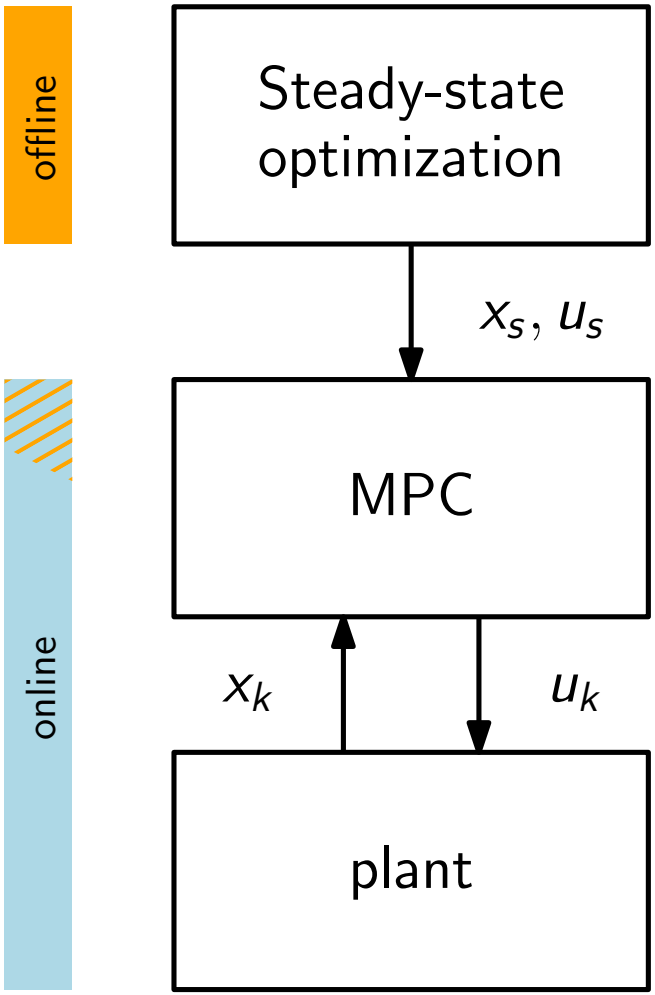
Let $\ell(x, u)$ be a **stage cost** that represent s economic losses, energy use, cost of material, yield, etc.

**MPC**

- $g(x_s, u_s) = 0$
- $g(x, u) \geq 0$
- $g(x, u) \in \mathcal{K}_\infty$
  continuous, zero at zero, strictly increasing, unbounded in $|\tilde{x}|$
- e.g., $g(x, u) = x^\top Q x + u^\top R u$

**Economic MPC**

- $\ell(x, u)$ continuous
- $\ell(x, u)$ lower-bounded in the domain
- e.g., linear functions and more

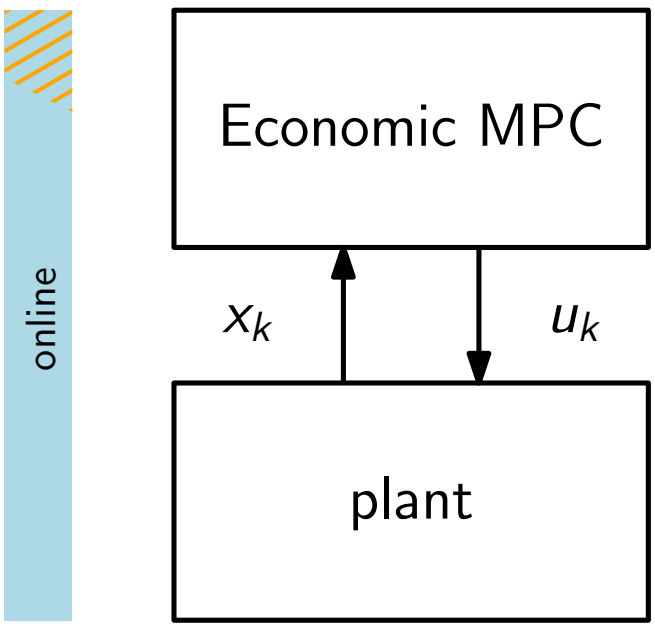



One solution is to represent this economic cost in the **steady-state optimization** problem, and to use the MPC cost to promote **tracking** of the steady state.

$$\begin{aligned} \min_{x_s, u_s} \quad & \ell(x_s, u_s) \\ \text{subject to} \quad & x_s = f(x_s, u_s) \\ & x \in \mathcal{X},\ u \in \mathcal{U} \end{aligned}$$

- steady-state optimization **offline** or at a **slower time-scale**
- suboptimal cost of the system trajectory
- cannot reduce cost during system transients
- does not respond **efficiently** to disturbances
- easy MPC real time optimization (linear-quadratic problem)

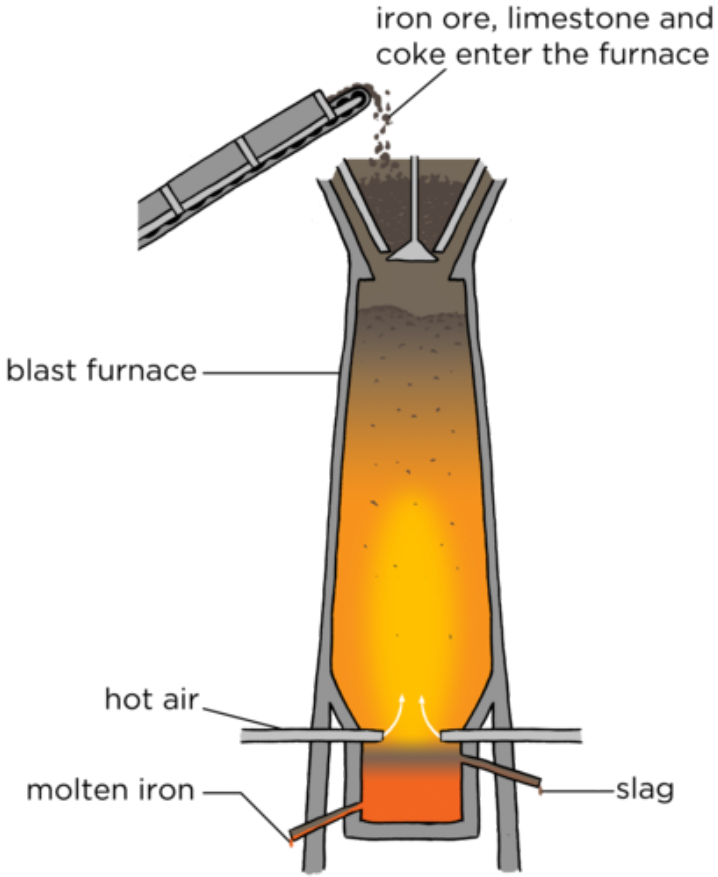
In **Economic MPC**, the loss function $\ell$ is directly used in the dynamic optimization!




$$
\begin{aligned}
V(x) = \min_{\mathbf{u},\mathbf{x}} \quad & \sum_{k=0}^{K-1} \ell(x_k, u_k) \\
\text{subject to} \quad & x_{k+1} = f(x_k, u_k), \quad x_0 = x \\
& x_k \in \mathcal{X} \\
& u_k \in \mathcal{U}
\end{aligned}
$$

- single time-scale
- the controller computes an efficient **trajectory**
- typically a more difficult optimization problem to solve in real-time
- particularly suited for nonlinear systems

# What could go wrong?




For some $(x, u)$ which are **not** steady states (equilibria) we can have

$$\ell(x, u) < \ell(x_s, u_s)$$

In fact, this is very common!

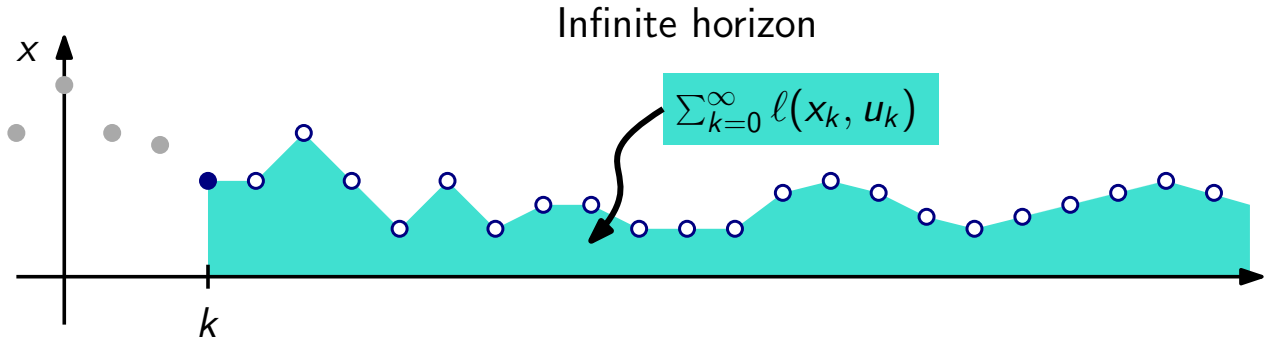
- $\approx$ furnace temperature
- $\downarrow$ inflow of iron ore / coke
- $\approx$ iron yield

$\Rightarrow$ furnace off... profit!?

## Convergence to steady state is not part of the specifications

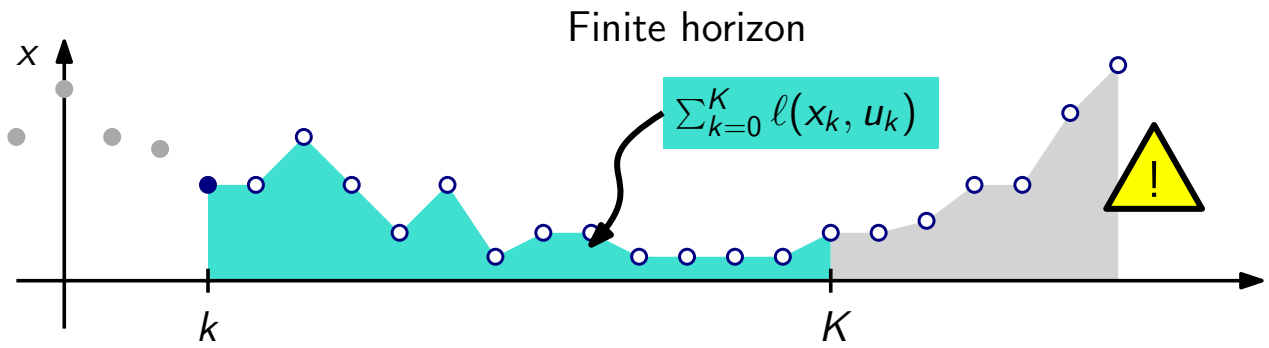
- \+ systems may be operated more efficiently away from equilibria (cycling, etc.)
- \- the design does not guarantee stability / boundedness of trajectories

# Infinite horizon



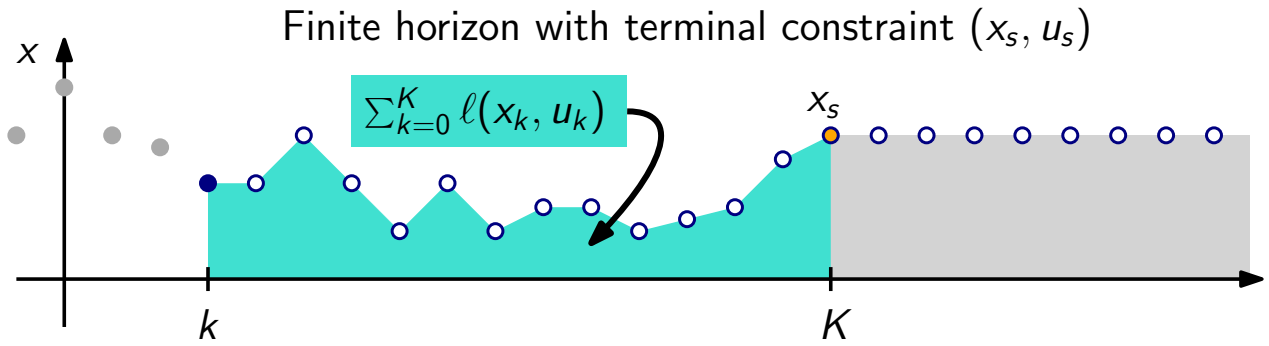

- Natural **economic interpretation**
  - continuous processes
  - no end in sight
  - interest in long-run economics
- The system may not be driven to an equilibrium
  - not a problem *per se*, if that is more efficient
- **Boundedness** is guaranteed by the constraints over the infinite trajectory
  - by construction, the controller produces an acceptable future trajectory
- **Very hard to solve computationally**
  - infinite-dimensional optimization problem

# Finite horizon




- **Computationally (more) tractable**
- Because we are not driving the system towards an equilibrium, we cannot assume that the "tail" is small (as we did with MPC tracking)
  - Finite horizon $\neq$ approximate infinite horizon
- Because low cost $\not\Rightarrow$ near equilibrium, **unstable trajectories** can emerge
  - cheap trajectory now, expensive fix later (outside the horizon)
- **Infeasibility** at the end of the control horizon can happen
  - Not all states are controllable
  - Input constraints may prevent recovery of the system

# Finite horizon with terminal constraint $(x_s, u_s)$




- **Computationally (more) tractable**
- **Future feasibility** is guaranteed
  - What is a feasible infinite-horizon trajectory?
- Closed-loop behavior
  - Is **closed-loop stability guaranteed?**
  - Is **the economic cost minimized?**

Remember that the closed-loop system does not realize the predicted trajectory

$$u_k = u^*(x_k)$$

# Economic MPC vs MPC

Consider the **linear system**

$$x_{k+1} = Ax_k + Bu_k, \qquad A = \begin{bmatrix} 1/2 & 1 \\ 0 & 3/4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

the **economic loss function**

$$\ell(x, u) = q^\top x + ru, \qquad q = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \qquad r = -10,$$

and the **constraints**

$$\mathcal{U} = [-1, 1]$$

## Optimal steady state

$$\begin{aligned} \min_{x_s, u_s} \quad & \ell(x_s, u_s) \\ \text{subject to} \quad & x_s = Ax_s + Bu_s \\ & x \in \mathcal{X},\ u \in \mathcal{U} \end{aligned}$$

Hints: where is the solution of a linear program?

# Economic MPC vs MPC

## MPC tracking

- Use $x_s^*, u_s^*$ as the desired target
- Define $\tilde{x} = x - x_s^*$ and $\tilde{u} = u - u_s^*$
- Use MPC to get there as quick as possible

$$g(\tilde{x}_k, \tilde{u}_k) = 10 \cdot \|\tilde{x}_k\|^2 + \|\tilde{u}_k\|^2$$

- Add terminal constraint $x_K = x_s^*$ to ensure stability
- In real time, solve the QP

$$\begin{aligned} \min_{\tilde{\mathbf{u}}, \tilde{\mathbf{x}}} \quad & \sum_{k=0}^{K-1} g(\tilde{x}_k, \tilde{u}_k) \\ \text{subject to} \quad & \tilde{x}_{k+1} = A\tilde{x}_k + B\tilde{u}_k \\ & \tilde{x}_K = 0 \end{aligned}$$

## Economic MPC

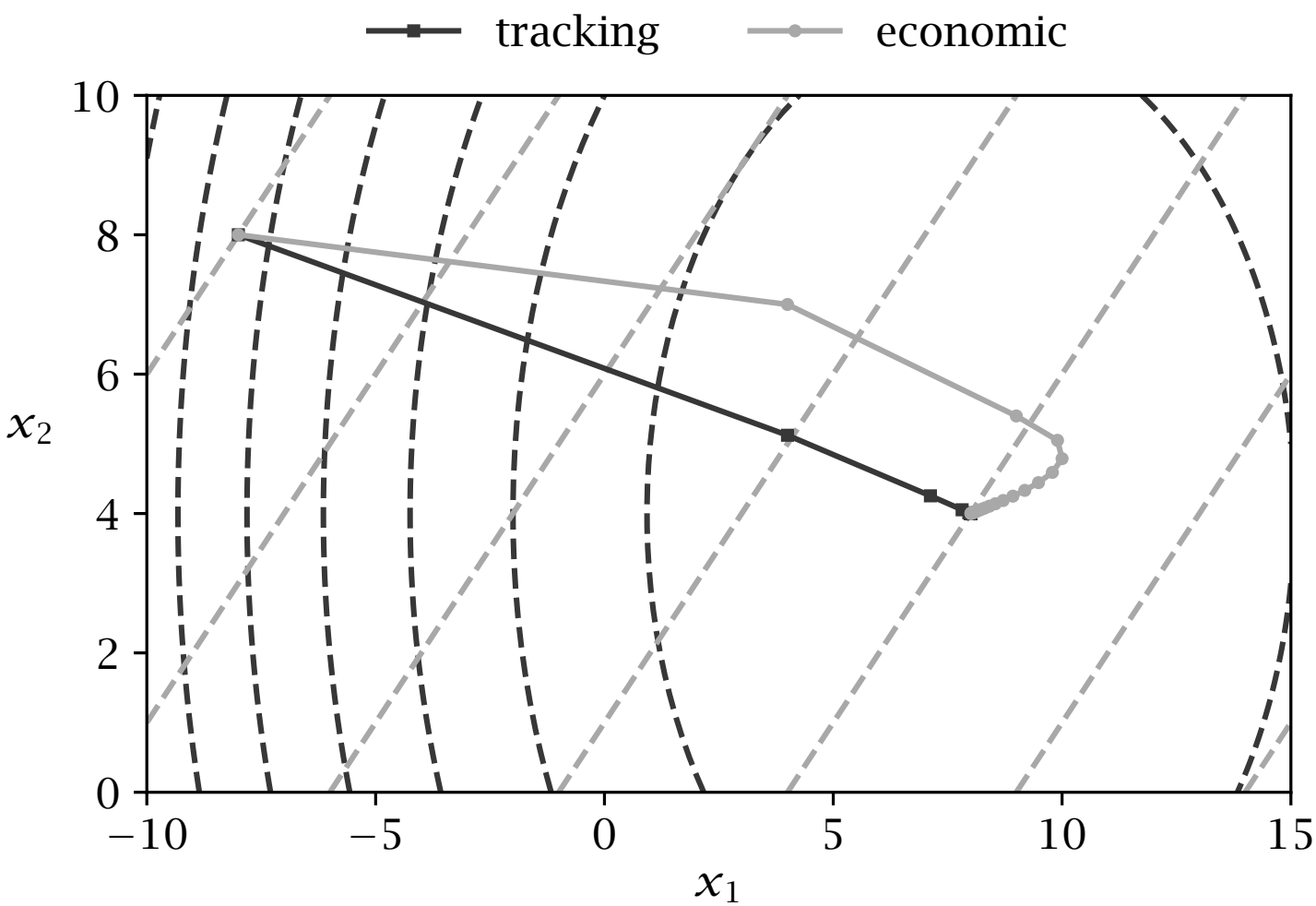
- Use directly $\ell(x, u)$ in the receding horizon controller

$$\ell(x_k, u_k) = q^\top x_k + r u_k$$

- Add terminal constraint $x_K = x_s^*$
- In real-time, solve the LP

$$\begin{aligned} \min_{\mathbf{u}, \mathbf{x}} \quad & \sum_{k=0}^{K-1} \ell(x_k, u_k) \\ \text{subject to} \quad & x_{k+1} = Ax_k + Bu_k \\ & x_K = x_S^* \end{aligned}$$

# Economic MPC vs MPC



- Closed-loop behavior
  - Is **closed-loop stability guaranteed?**
  - Is **the economic cost minimized?**

Let's answer these questions.

# Stability of Economic MPC

In **tracking MPC**, the terminal constraint $x_K = x_s$ was essential in proving that

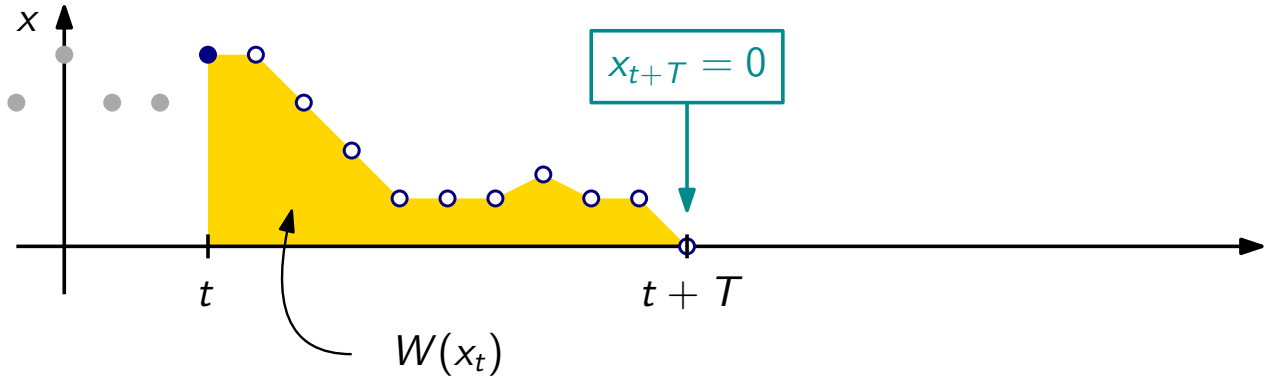
$$
\begin{aligned}
W(\tilde{x}) = \quad & \min_{\tilde{\mathbf{u}},\tilde{\mathbf{x}}} \quad \sum_{k=0}^{K-1} g(\tilde{x}_k, \tilde{u}_k) \\
& \text{subject to} \quad \tilde{x}_{k+1} = f(\tilde{x}_k, \tilde{u}_k) \\
& \qquad\qquad\quad \tilde{x}_0 = \tilde{x},\ \tilde{x}_K = 0
\end{aligned}
$$

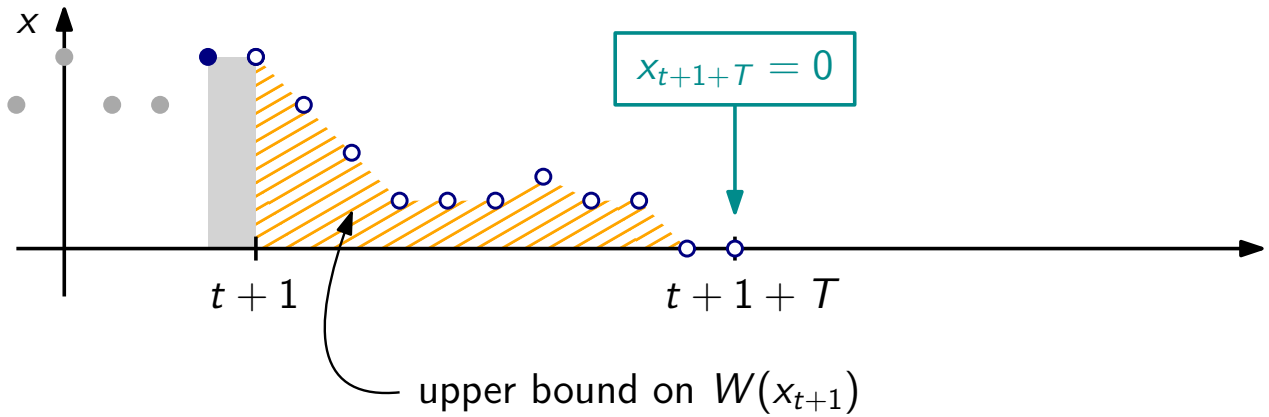
is a valid **Lyapunov function** for the plant.

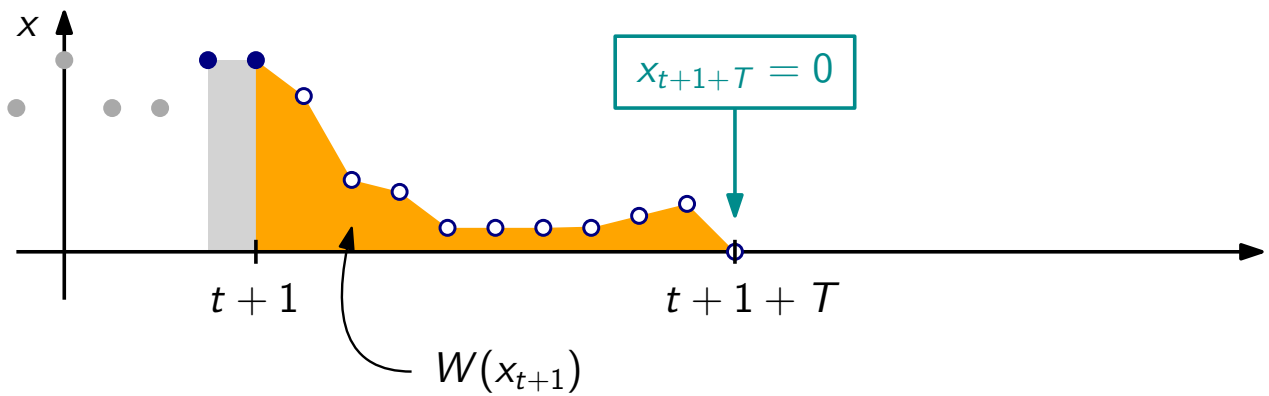
## Key idea:

$$
W(f(\tilde{x}, u_0^*(\tilde{x})) \leq W(\tilde{x}) + \underbrace{g(0,0)}_{=0} - g(\tilde{x}, u_0^*(\tilde{x})) \leq W(\tilde{x})
$$

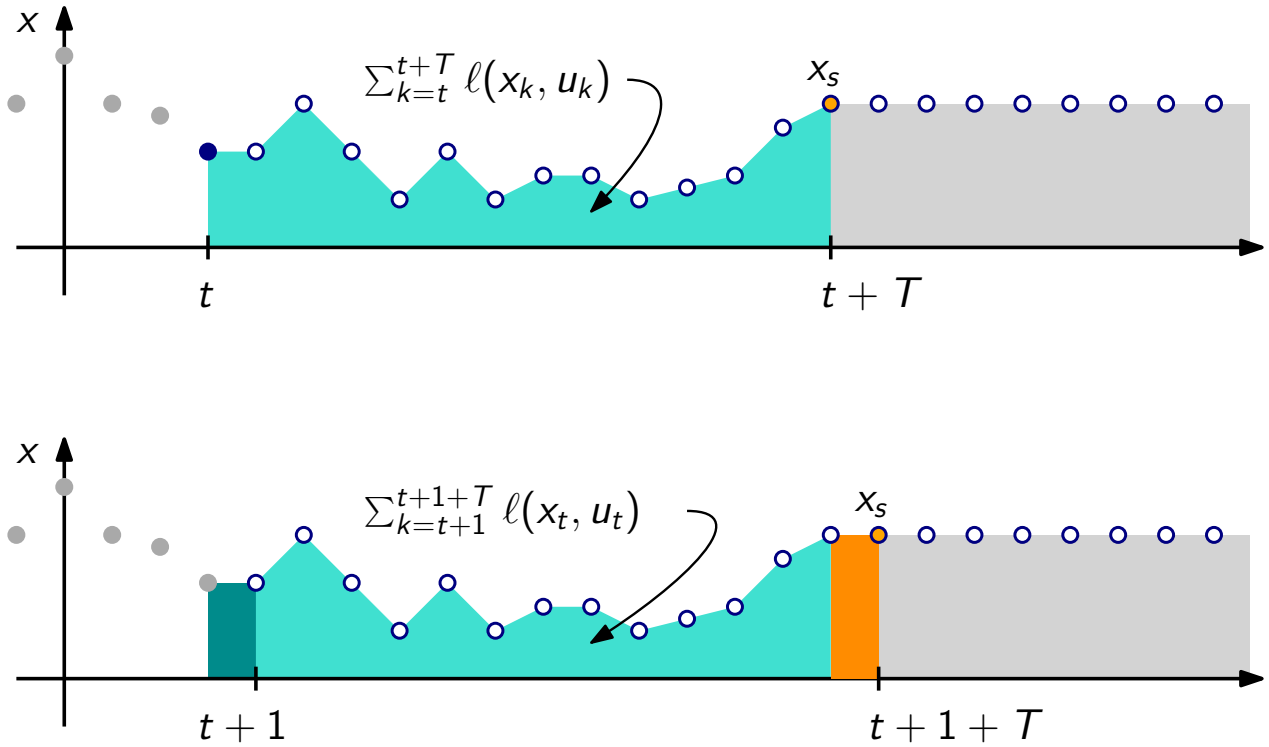
because a valid new trajectory was to keep the state at $\tilde{x} = 0$ at the new "tail".


x
$x_{t+T} = 0$
$t$
$t + T$
$W(x_t)$


x
$x_{t+1+T} = 0$
$t + 1$
$t + 1 + T$
upper bound on $W(x_{t+1})$


x
$x_{t+1+T} = 0$
$t + 1$
$t + 1 + T$
$W(x_{t+1})$

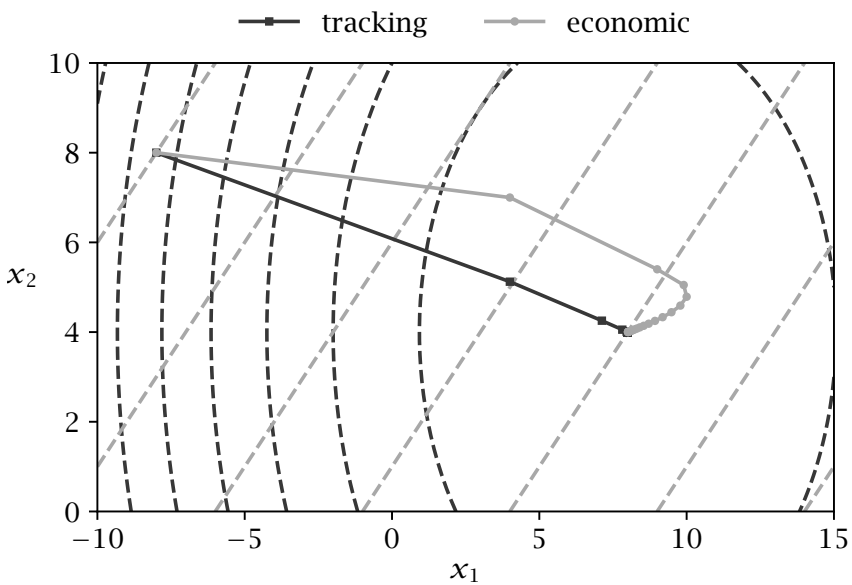
# Can we apply the same trick to Economic MPC?

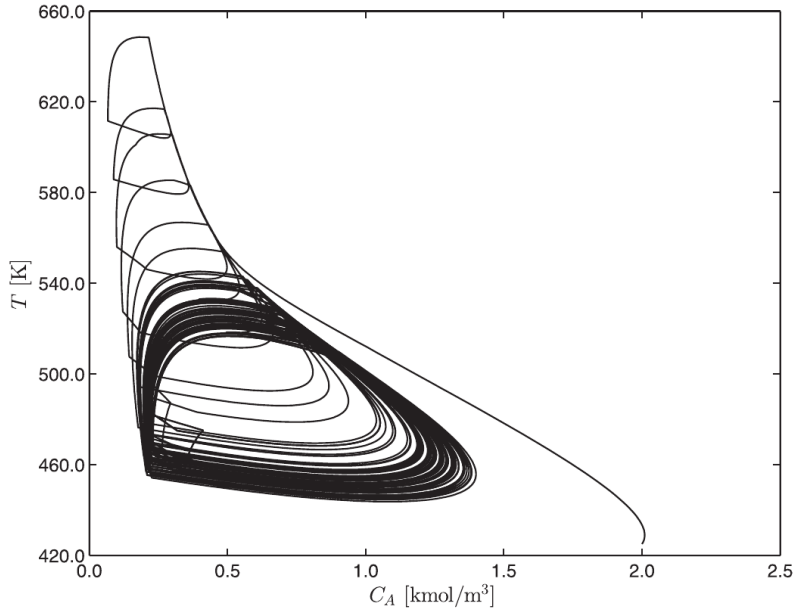



$$W(f(x, u_0^*(x)) \leq W(x) + \ell(x_s, u_s) - \ell(x, u_0^*(x)) \overset{??}{<} W(x)$$

In general, no. $\ell$ is not minimized at the steady state!
It may be "expensive" the keep the system at equilibrium.

Stronger conditions are required to guarantee **asymptotic stability**.




- Is asymptotic stability of the steady state really desireable?
- Can I achieve lower cost by **not** converging to an equilibrium?
- Are there outperforming periodic solutions?

# Asymptotic average performance

## Theorem

Let $x_k$ be the closed-loop trajectory generated by the Economic MPC controller with terminal constraint $x_s$. Then, for any initial condition,

$$\limsup_{t\to\infty} \sum_{k=0}^{t-1} \frac{\ell(x_k, u_k)}{t} \leq \ell(x_s, u_s).$$

Because of the terminal constraint, we have that the optimal cost over a finite horizon satisfies

$$W(f(x, u_0^*(x)) \leq W(x) - \ell(x, u_0^*(x)) + \ell(x_s, u_s)$$

Sum over $t$ steps and divide by $t$ to get the average performance over $t$ time steps:

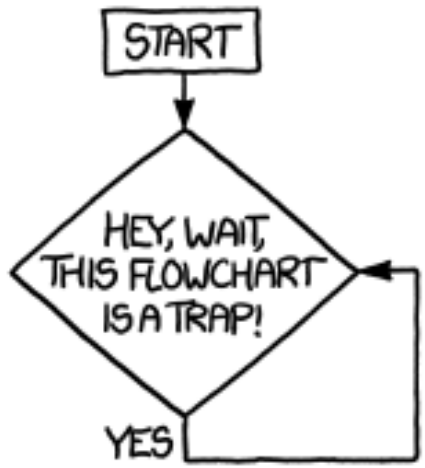
$$\sum_{k=0}^{t-1} \frac{\ell(x_k, u_k)}{t} \leq \ell(x_s, u_s) + \frac{1}{t}\left(W(x_0) - W(x_t)\right)$$

Note that $W(x_1), W(x_2), \ldots, W(x_{t-1})$ terms cancel out. We take the limit and as $t \to \infty$, the term $\frac{W(x_0)}{t} \to 0$. We make use of the boundedness (below) of $W$ (follows from boundedness of $\ell(x, u)$) to conclude the proof. Note that $W$ is not a Lyapunov function here.

# In summary

- A **terminal constraint** is necessary in order to maintain feasibility of the EMPC problem.
- EMPC may produce trajectories that are efficient and economical and **do not converge to an equilibrium of the system**, such as limit cycles/oscillations in nonlinear plants.
- There exists additional conditions to make the steady state **asymptotically stable** for EMPC (not covered in this lecture),
- Even in the case in which the system converges to the steady state, EMPC produces **efficient trajectories to the equilibrium**
  - trajectories are provably more efficient than the steady state over a long enough horizon
  - this advantage may accumulate in case of frequent disturbances that displace the system from the equilibrium

## The control engineer flowchart





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