

Power Flow Equations

Saverio Bolognani, ETH Zurich

EECI School M07, May 6-10, 2024, Barcelona

Outline

- The “direct” PF problem
- The “inverse” power flow problem
- Existence of PF solutions for PQ networks
- A geometric interpretation

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“Direct” problem

Given bus voltages $\vec{v}$, it is always possible to compute power flows $\vec{s}$ such that

$$\vec{s} = \operatorname{diag}(\vec{v})\ (Y\vec{v})' \qquad Y = G + jB + Y^{\text{shunt}}$$

where G, B are real-valued Laplacians, Y^{shunt} diagonal.

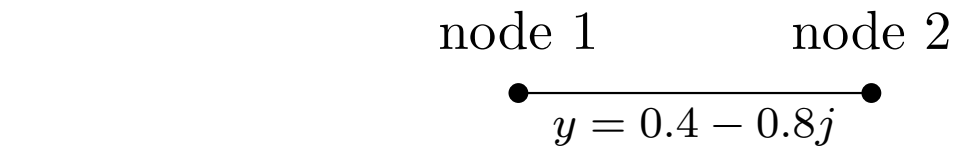
Examples

1. Flat voltage solution: $\vec{v} = \mathbb{1}$, no shunt admittances
   - often used in stability analysis (as the equilibrium of generator dynamics)
2. Flat voltage solution: $\vec{v} = \mathbb{1}$, shunt line capacitance
3. Constant angle: $\vec{v} = v \in \mathbb{R}$, inductive lines, no shunt
4. Constant magnitude: $\vec{v} = e^{j\theta}$, inductive lines, no shunt, small θ

1. $\vec{s} = \operatorname{diag}(\mathbb{1})\ (Y\mathbb{1})' = I \cdot \mathbb{0} = \mathbb{0}$
2. $\vec{s} = \operatorname{diag}(\mathbb{1})\ (j(B + \omega C)\mathbb{1})' = -j\omega C\mathbb{1}$
3. $\vec{s} = \operatorname{diag}(v)\ (jBv)' = -j \operatorname{diag}(v)Bv$
4. $\vec{s} \approx \operatorname{diag}(\mathbb{1} + j\theta)\ (jB(\mathbb{1} + j\theta))' = -(I + j \operatorname{diag}(\theta))B\theta \approx -B\theta + o(\theta^2)$

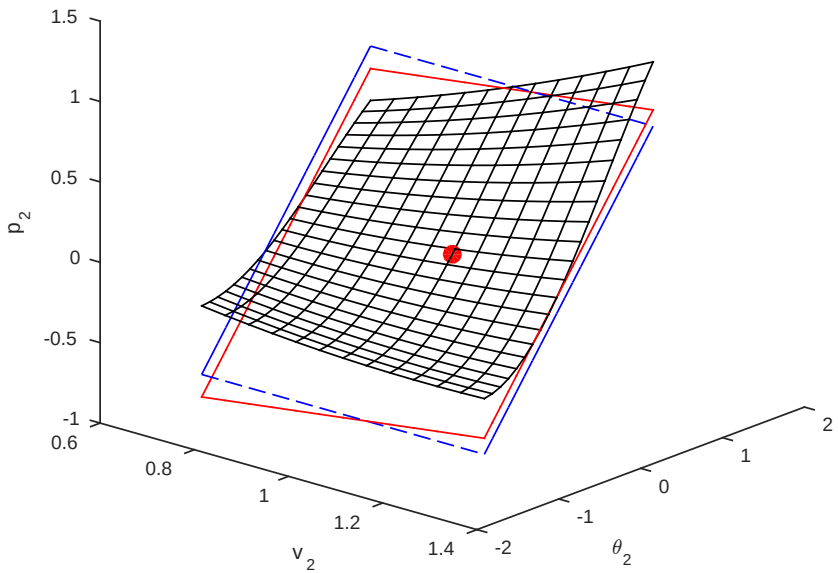
“Direct” problem: numerical example

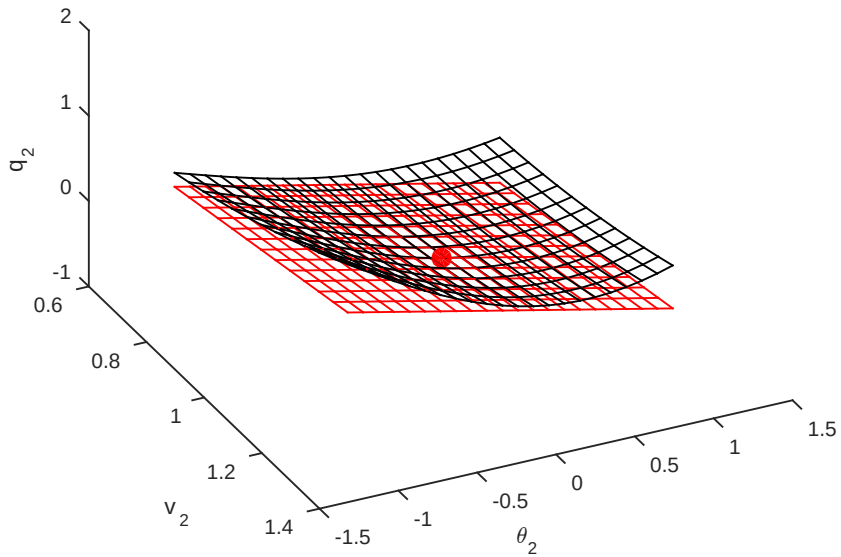
Simplest grid: two-bus network, generic line impedance, no shunt admittance



$v_1 = 1,\ \theta_1 = 0$ $\quad$ $v_2,\ \theta_2$

$p_1,\ q_1$ $\quad$ $p_2,\ q_2$

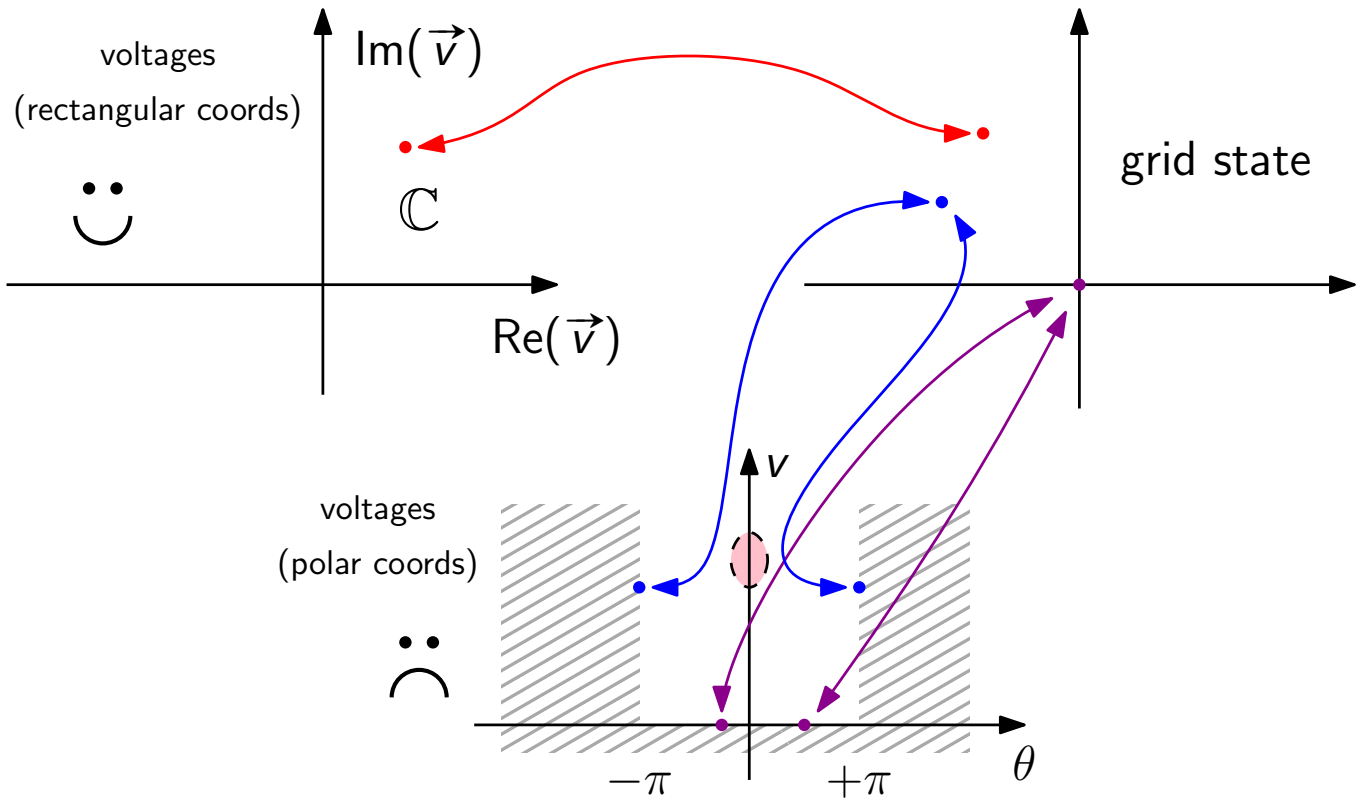




Homeomorphism to $\mathbb{C}^N$

Lesson learned from the direct problem

The state of a N-bus grid is homeomorphic to $\mathbb{C}^N$ (or $\mathbb{R}^{2N}$)
$\rightarrow$ there exists a bijective map between the **set of grid states** and $\mathbb{C}^N$



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“Inverse problem”

Specifications are typically given in terms of **nodal power injections**

- power delivered to a load bus
- power injected by a generator

That’s the purpose of a power system!

Load flow problem

Find the grid state $\vec{v}$ that yields the desired bus power injections $\vec{s}$
$\rightarrow$ valid set-points for the low-level controllers

$$\vec{s} = \mathrm{diag}(\vec{v})\ (Y\,\vec{v})'$$

Much more complicated problem:

- rotational symmetry
- multiple/zero solutions
- numerically ill-posedness

Rotational symmetry

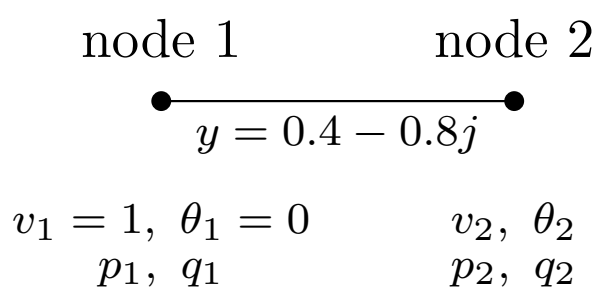
Uniform phase shift ψ of the voltages does not affect the power flows:

$$\operatorname{diag}\left(\vec{v}e^{j\psi}\right)\left(Y\vec{v}e^{j\psi}\right)' = \operatorname{diag}\left(\vec{v}\right)\left(Y\vec{v}\right)'$$

Physical interpretation

Uniform phase shift is just an arbitary choice of t_0 for the oscillator

$$\sin(\omega(t-t_0)+\theta)$$



Reference bus

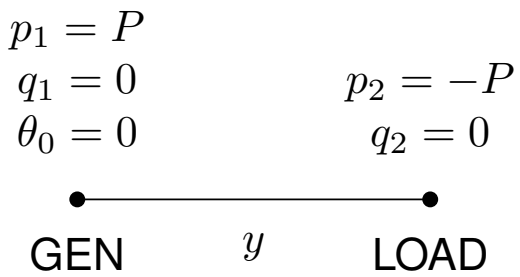
$$\theta_1 = 0$$

($2N-1$ d.o.f. for the grid state)

Careful with:

- dynamic analysis (infinite inertia?)
- with PMUs (absolute t)

Multiple/zero solutions



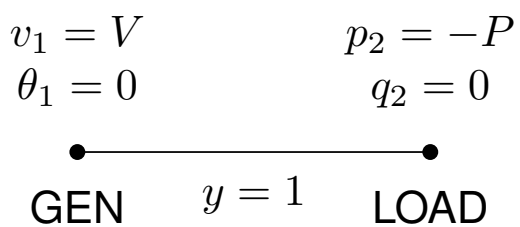
Example 1: Specifications

- Deliver power P to LOAD
- Generate power P from GEN

Is the problem feasible?

$0 = \mathbb{1}^\top \vec{s} = \mathbb{1}^\top \operatorname{diag}(\vec{v})(Y\vec{v})' = \vec{v}^T G v' + j\,\vec{v}^T B \vec{v}' \;\Rightarrow\; \vec{v} = \alpha \mathbb{1}$

No power flow!



Example 2: Specifications

- Deliver power $-P_1$ to LOAD
- GEN voltage V_0 (slack bus)

Is the problem feasible?

Example: 2-bus case

For 2 buses, the PF equations can be solved analytically.

$$\vec{v}_2 \left((\vec{v}_2 - \vec{v}_1)\, y \right)' = \vec{s}_2$$

By using the specifications $y = 1$, $v_1 = V$, $\vec{s}_2 = -P$ we get

$$v_2^2 - v_2 e^{j\theta_2} V + P = 0 \quad \Rightarrow \quad v_2 V \sin\theta_2 = 0 \quad \rightarrow \quad \theta_2 = 0$$

and therefore we obtain the quadratic equation $v_2^2 - V v_2 + P = 0$.

- If $V^2 < 4P$, there are no solutions
- If $V^2 > 4P$, there are two distinct solutions

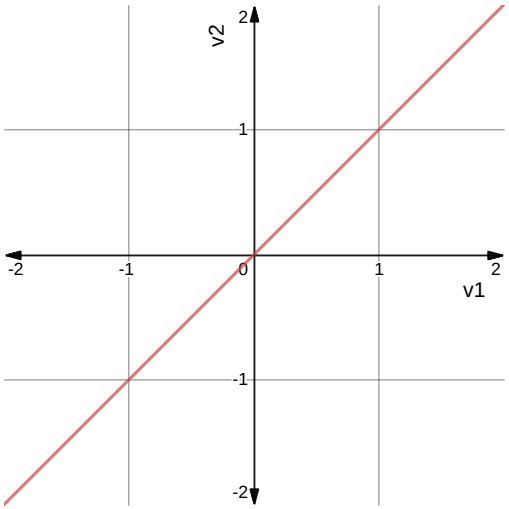
$$v_2 = \frac{V}{2} \pm \frac{\sqrt{V^2 - 4P}}{2}.$$

Notice that, if V is large, then the solutions exist and, since $\sqrt{V^2 - 4P} = V\sqrt{1 - 4P/V^2} \simeq V(1 - 2P/V^2) = V - 2P/V$, the two solutions are approximated by $V - \frac{P}{V}$ and $\frac{P}{V}$.
At these solutions, the current i_2 takes (respectively) the values $i_2 \simeq -P/V$ and $i_2 \simeq V$.
Therefore, when V is large, one solution consists in small currents (thus small power losses and voltage close to the nominal voltage across all the network), while the other consists in larger currents, larger power losses, and larger voltage drops.

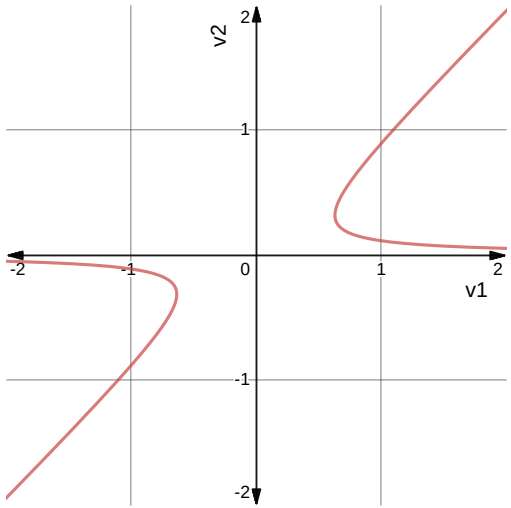
Of course, the system should work at the first working point.

2-bus case: nose curve

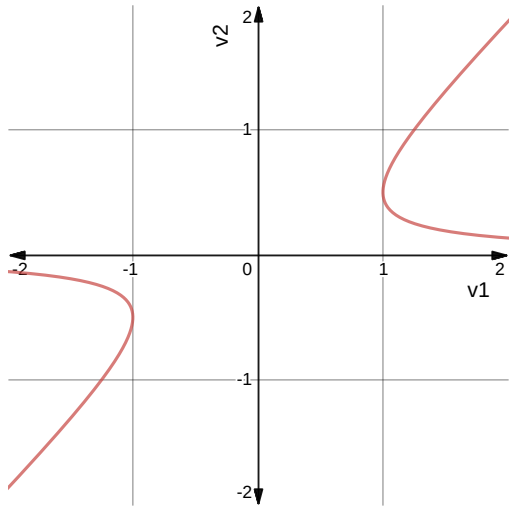
$P = 0$



$P = 0.1$

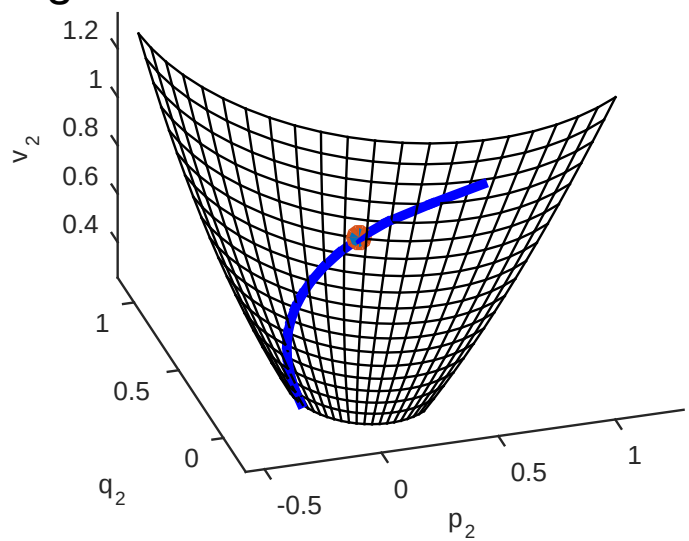


$P = 0.25$



Nose curve (blue): 1-D curve on the set of "valid" grid states.

- "inverse" problem not always feasible
- "inverse" problem not always well-posed
- **power feasible region**
  $\vec{s}$ for which a valid grid state exist



Very classic problem in power system analysis

Extension to power network

How does this extend to general power networks?

Typical specifications

- General grid topology
- Nodal constraints depending on which quantities are specified
  - **PQ buses**: $\vec{s}_i = p_i + jq_i$ is specified
    - loads
    - generators whose local controllers can track PQ references
    - "dummy" buses $\vec{s}_i = 0$
  - **PV buses**: p_i and v_i are specified
    - generators whose local controllers can track PV references
  - **slack buses**: $\vec{v}_i = v_i e^{j\theta_i}$ is specified
    - bus that serves as a **reference bus** and ensures power balance

Does a steady-state that satisfy the given specifications exist?

The answer is in general not trivial.

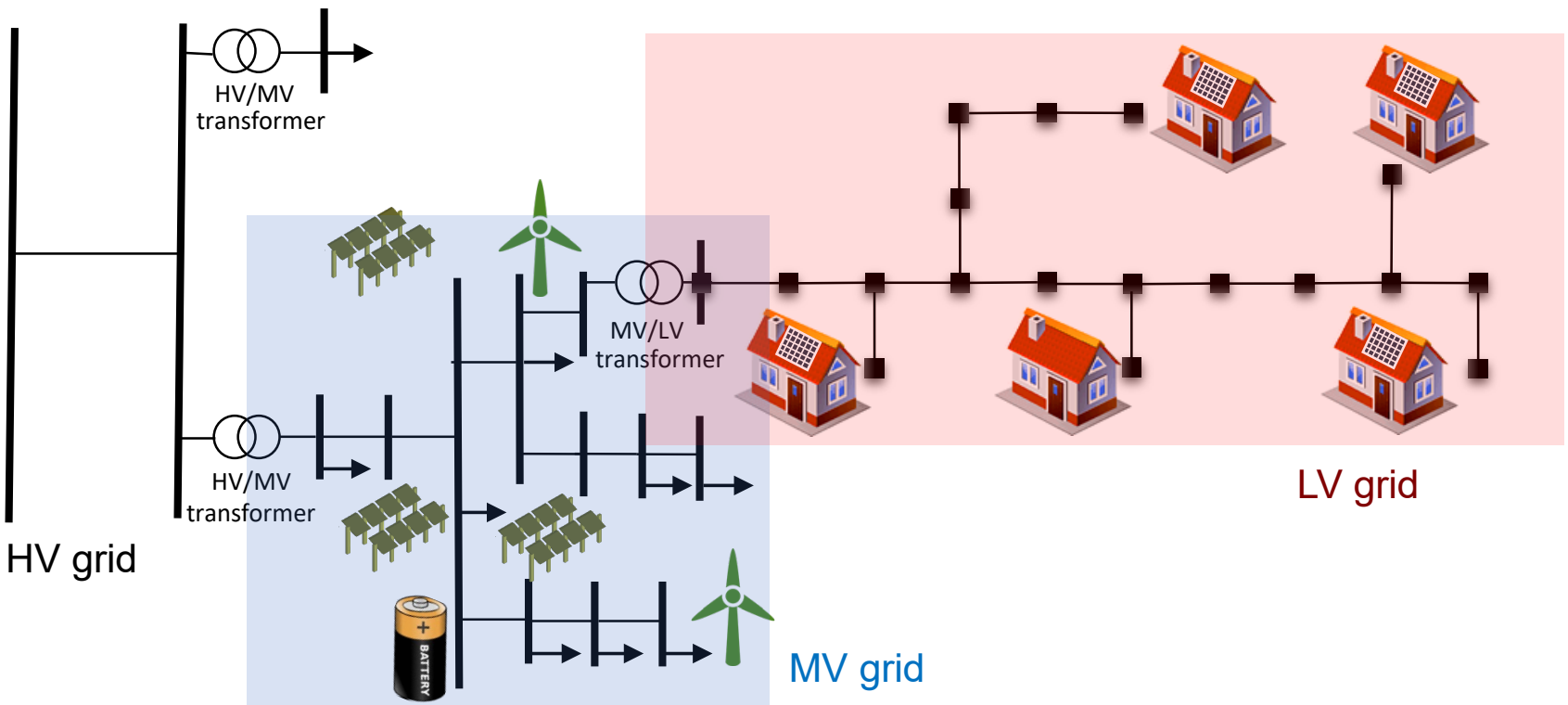
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Extension to networks of PQ buses

Special case

N-bus grid with 1 slack bus, $N-1$ PQ buses



Z-bus matrix

Consider a network of PQ buses, where bus 1 is the slack bus ($v_1 = V$). Assume zero shunt elements.

Z-bus matrix

We define the Z-bus matrix (bus impedance matrix) as the pseudoinverse of the bus admittance matrix Y, i.e.,

$$Z = Y^\dagger = Z \mid ZY = I - \mathbb{1}\mathbb{e}_1^\top, Z\mathbb{e}_1 = 0$$

- $\vec{v} = V\mathbb{1} + Z\vec{i}$
  Left-multiply $\vec{i} = Y\vec{v}$ by Z. We get

$$Z\vec{i} = ZY\vec{v} = v - \mathbb{1}\mathbb{e}_1^\top v = v - \mathbb{1}V$$

- How to construct Z

$$\begin{bmatrix} Z & \mathbb{1} \\ \mathbb{1}^\top & 0 \end{bmatrix} \begin{bmatrix} Y & \mathbb{e}_1 \\ \mathbb{e}_1^\top & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} Z & \mathbb{1} \\ \mathbb{1}^\top & 0 \end{bmatrix} = \left(\begin{bmatrix} Y & \mathbb{e}_1 \\ \mathbb{e}_1^\top & 0 \end{bmatrix} \right)^{-1}$$

Z-bus matrix

- Z is positive semi-definite, symmetric
- What is Z_{ii}?
  $\mathbb{e}_i^T Z \mathbb{e}_i = \mathbb{e}_i^\top (\vec{v} - V\mathbb{1})\mathbb{e}_i$ when $\vec{i} = \mathbb{e}_i$
  $\Rightarrow$ effective resistance between the slack bus and bus i.
- What is Z_{ij} in a radial network?
  Z_{ij} is the impedance of the common part of the path between node i and node 1 and between node j and node 1.

Extension to networks of PQ buses

S. Bolognani and S. Zampieri
On the existence and linear approximation of the power flow solution in power distribution networks
IEEE Transactions on Power Systems, 31(1):163-172, 2016. → preprint

$\star$ -norm

$\|Z\|^\star := \max_h \|Z_{h\bullet}\|$ (largest row norm).

$$\min_{\substack{\|x\|=1\\ \|y\|=1}} \|\mathrm{diag}(x) Z y\| = \|Z\|^\star$$

Theorem

Let

$$\|\vec{s}\| < \frac{1}{4} S \qquad \text{where} \qquad S := \frac{V^2}{\|Z\|^\star}.$$

Then there exists a unique solution

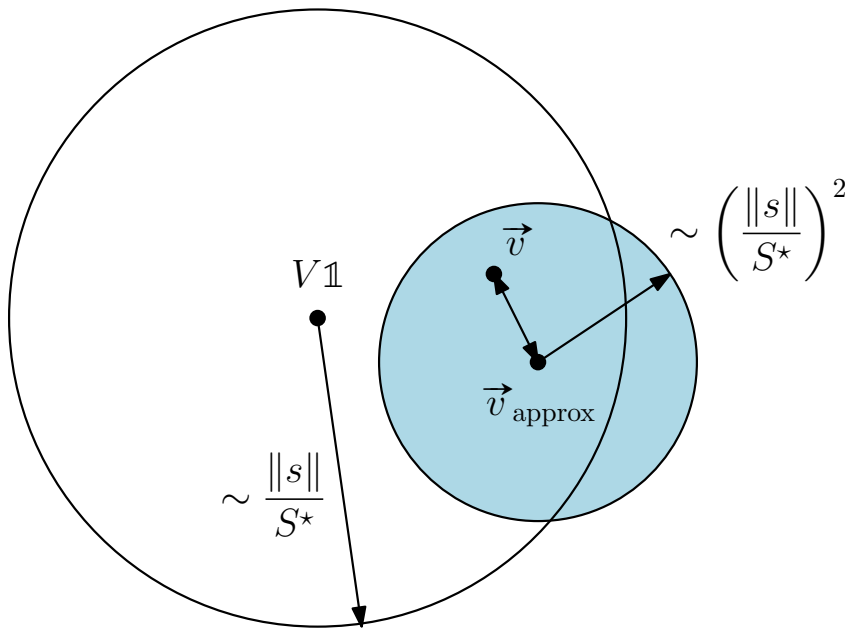
$$\vec{v} = V\left(\mathbb{1} + \frac{Z}{\|Z\|^\star}\frac{\vec{s}\,'}{S} + \frac{Z}{\|Z\|^\star}\vec{\lambda}\right) \qquad \text{that satisfies} \qquad \|\vec{\lambda}\| < 4\left(\frac{\|\vec{s}\|}{S}\right)^2$$

Intepretation

$$\vec{v} = \underbrace{V\left(\mathbb{1} + \frac{Z}{\|Z\|^\star}\frac{\vec{s}'}{S}\right)}_{\vec{v}_{\text{approx}}} + \frac{Z}{\|Z\|^\star}\vec{\lambda}\Bigg) \qquad \text{with} \qquad \|\vec{\lambda}\| < 4\left(\frac{\|\vec{s}\|}{S}\right)^2$$

Dual role of the error term

- **upper bound** on the quality of the approximation
- **lower bound** on the neighborhood where no other solutions exist



Change of coordinates

Remember that $v = V\mathbb{1} + Zi$ and define x as

$$x := Vi' - s \qquad \text{(interpretation: } x = 0 \text{ if } v = V\mathbb{1}\text{)}$$

Therefore

$$i = \frac{1}{V}(s + x)'$$

and then

$$\begin{aligned} s = \operatorname{diag}(i')v &= \frac{1}{V}\operatorname{diag}(s + x)(V\mathbb{1} + Zi) \\ &= s + x + \frac{1}{V}\operatorname{diag}(s + x)Zi \\ &= s + x + \frac{1}{V^2}\operatorname{diag}(s + x)Z(s + x)' \end{aligned}$$

Power flow eqs in $x - s$ coordinates

$$x = \underbrace{-\frac{1}{V^2}\operatorname{diag}(s + x)Z(s + x)'}_{:=G(x)} \qquad \text{together with} \quad v = V\mathbb{1} + \frac{1}{V}Z(s + x)'$$

Sketch of the proof

Banach's Fixed Point Theorem

Let $(\mathcal{X}, d)$ be a complete metric space. Let $G : \mathcal{X} \to \mathcal{X}$ be a **contraction map** on $\mathcal{X}$, i.e., there exists $\alpha \in [0, 1)$ such that

$$d(G(x), G(y)) \leq \alpha d(x, y)$$

for all x, y in $\mathcal{X}$. Then G admits a **unique fixed-point** $x^\star \in \mathcal{X}$.

Key steps

1. Impose $\|G(x) - G(y)\| < \|x - y\|$ (contraction)
2. Obtain conditions on $\|s\|$ where this is true
3. Find a small invariant set $\mathcal{X} := \{x \,|\, \|x\| < \delta\}$ where these conditions are satisfied.
4. Use Banach's Fixed Point Theorem to conclude on the existence (and uniqueness) of a fixed point of $G(x)$, which is a solution of the power flow equations.

Sketch of the proof

$$\|G(x) - G(y)\| \leq \ldots \leq \frac{1}{V^2}\|Z\|^\star \left(\|x+y\| + 2\|s\|\right)\|x-y\|$$
$$\leq \underbrace{\frac{2}{V^2}\|Z\|^\star(\delta + \|s\|)}_{\alpha}\|x-y\| \quad \text{when} \quad \|x\|, \|y\| \leq \delta$$

In order to make $\alpha < 1$ we set

$$\frac{2}{V^2}\|Z\|^\star\|s\| < \frac{1}{2} \quad \Rightarrow \quad \|s\| < \frac{1}{4}\frac{V^2}{\|Z\|^\star} \tag{1}$$

and

$$\frac{2}{V}\|Z\|^\star\delta < \frac{1}{2} \tag{2}$$

We look for a δ that, under the assumption (1), satisfies (2) and $\|G(x)\| < \delta$.

One valid example is $\delta = \frac{4\|Z\|^\star}{V^2}\|s\|^2$ (verify).

Another example is $\delta = \|s\|$, but remember that we are trying to find a small neighborhood to show accuracy of the linear approximant.

We therefore apply Banach Fixed Point Theorem and conclude that there exists a unique x in

$$\mathcal{X} = \left\{ x \;\middle|\; \|x\| < \frac{4\|Z\|^\star}{V^2}\|s\|^2 \right\}$$

such that $G(x) = x$ (solution of the PF equations).

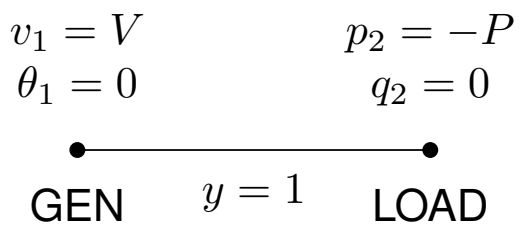
We then have

$$\begin{aligned} v &= V\left(\mathbb{1} + \frac{1}{V^2} Z(s+x)'\right) \\ &= V\left(\mathbb{1} + \frac{Z}{\|Z\|^\star}\frac{s'}{S} + \frac{Z}{\|Z\|^\star}\underbrace{\frac{x'}{S}}_{\lambda}\right) \end{aligned}$$

where

$$\|\lambda\| = \left\|\frac{x'}{S}\right\| < \frac{1}{S}\frac{4\|Z\|^\star}{V^2}\|s\|^2\frac{1}{S} = \frac{\|s\|^2}{S^2}.$$

“Sanity check”



$$Z = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\|Z\|^\star = 1, S = V^2$

$\|\vec{s}\| = P$

The theorem says that if $\|\vec{s}\| < S/4 = V^2/4$ then

$$\vec{v}_2 = V\left(1 + \frac{1}{V^2}(-P) + \lambda_2\right) \quad \text{with} \quad \|\vec{\lambda}_2\| < 4\|s\|^2/S^2 = 4P^2/V^4.$$

Analytic bound for existence: $V^2 > 4P$

Analytic solution: $v_2 = \frac{V}{2} + \frac{\sqrt{V^2 - 4P}}{2}$

The theorem’s condition on V and the bound on $\|\vec{\lambda}\|$ are tight!

At $P = V^2/4$ we get $v_2 = V/2$ and therefore

$$\frac{V}{2} = V\left(1 - \frac{1}{4} + \lambda_2\right) \quad \Rightarrow \quad \lambda_2 = -\frac{1}{4} \qquad \|\lambda_2\| \leq \frac{4P^2}{V^2} = \frac{1}{4}$$

Interpretation of $\|s\| < S/4$

It is easy to prove that $\|Z\|^\star \leq \|Z\|$, and therefore (under minor assumptions) $\|Z\|^\star \leq \frac{1}{\sigma_2(Y)}$ where $\sigma_2(Y)$ is the **algebraic connectivity** of the grid.

Bound based on algebraic connectivity

$$\|s\| \leq \frac{1}{4} V^2 \sigma_2(Y).$$

Same criterion in generic norms $p, q \in [1, \infty]$:

$$\|s\|_p < \frac{1}{4} S \qquad \text{where} \qquad S := \frac{V^2}{\|Z\|_q^\star} \qquad \text{with} \qquad \frac{1}{p} + \frac{1}{q} = 1$$

$p = 1, q = \infty$

$\|s\|_1 := S_{\text{tot}}$ total power demand of the grid
$\|Z\|_\infty := \ell_{\text{max}}$ "breadth" of the grid (longest distance from node 1)

$$S_{\text{tot}} < \frac{1}{4} \frac{V^2}{\ell_{\text{max}}}$$

Further steps

Generalization based on the same fixed-point argument

C. Wang, A. Bernstein, J. Le Boudec, M. Paolone
Explicit Conditions on Existence and Uniqueness of Load-Flow Solutions in Distribution Networks
IEEE Transactions on Smart Grid, 9(2), 2018.

A. Bernstein, C. Wang, E. Dall'Anese, J. Le Boudec, C. Zhao
Load Flow in Multiphase Distribution Networks: Existence, Uniqueness, Non-Singularity and Linear Models
IEEE Transactions on Power Systems, 33(6), 2018.

How to make the criterion less conservative

S. Yu, H. D. Nguyen, K. S. Turitsyn
Simple certificate of solvability of power flow equations for distribution systems
2015 IEEE Power & Energy Society General Meeting

Numerical construction of power flow feasibility sets

K. Dvijotham, K. Turitsyn
Numerical method to construct the power feasible region
arXiv prepring

From fixed-point to numerical solvers

M. Bazrafshan, N. Gatsis
Convergence of the Z-Bus Method for Three-Phase Distribution Load-Flow with ZIP Loads
IEEE Transactions on Power Systems, 33(1), 2018)

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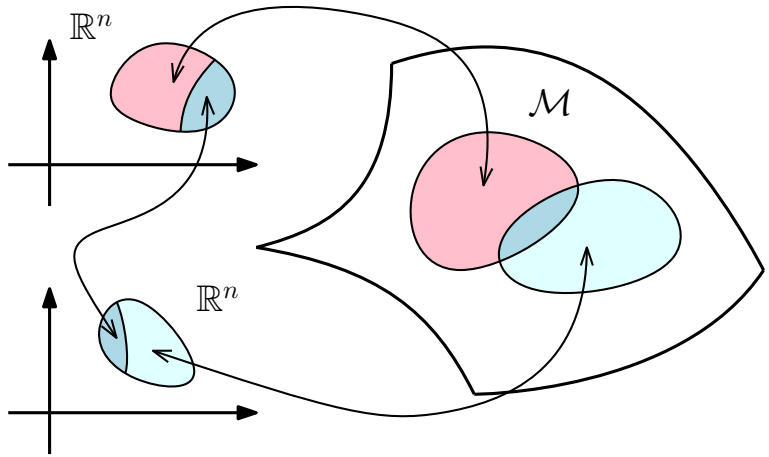
Preliminaries

Smooth n-dimensional manifold

A smooth n-dimensional manifold $\mathcal{M}$ is a set of points that around every point $x \in \mathcal{M}$ is locally Euclidean. That is, around x, $\mathcal{M}$ "looks like" $\mathbb{R}^n$.

Chart: map from a domain $\mathcal{U} \subset \mathcal{M}$ to a subset of $\mathbb{R}^n$ and viceversa.

A set of charts that cover the entire manifold and which are "compatible" form an **atlas** (i.e., there is a smooth coordinate transformation between any two chart domains that overlap)



Embedded manifold

Let $F : \mathbb{R}^m \to \mathbb{R}^n$, with $\nabla F(x)$ of rank $k \leq n$ everywhere. Then

$$\mathcal{M} = \{x \mid F(x) = r\}$$

is a manifold of dimension $m - k$.

A geometric interpretation of the PF eqs

Power flow manifold (PFM)

The set of all steady states of a n-bus power system that satisfy the grid power flow equations are a manifold of dimension $2n$.

We already know a **global chart**: remember the "direct" PF problem?

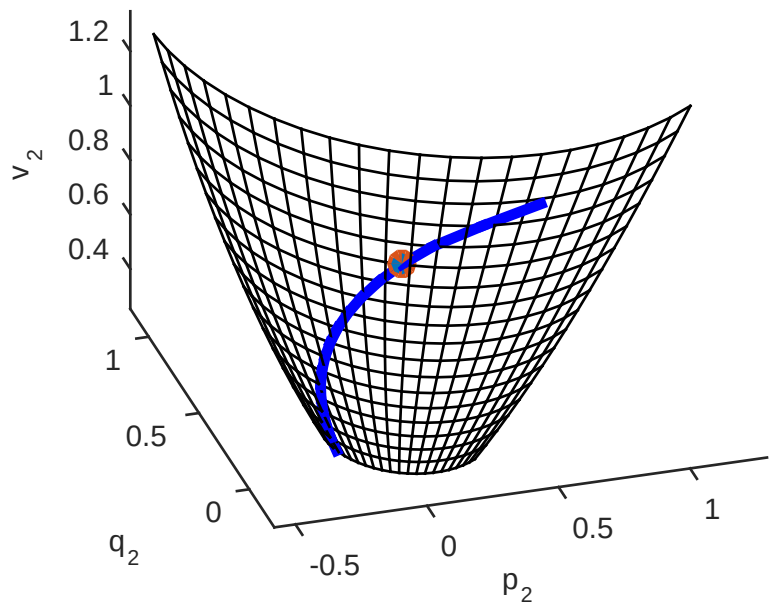
Embedding of the PFM in $\mathbb{R}^{4n}$

$$F(v,\theta,p,q) := \begin{bmatrix} \Re\left[\mathrm{diag}(v)(Yv)'\right] - p \\ \Im\left[\mathrm{diag}(v)(Yv)'\right] - q \end{bmatrix} = 0$$

$F : \mathbb{R}^{4n} \to \mathbb{R}^{2n}$ and

$$\nabla F = \begin{bmatrix} \nabla_{v,\theta} F & -I_{2n} \end{bmatrix}$$

has rank $2n$
$\to$ smooth manifold of dimension $2n$.



Remark

The PFM is an **abstract object** that is not defined by a specific chart (i.e., by a specific set of coordinates)!

Alternatives

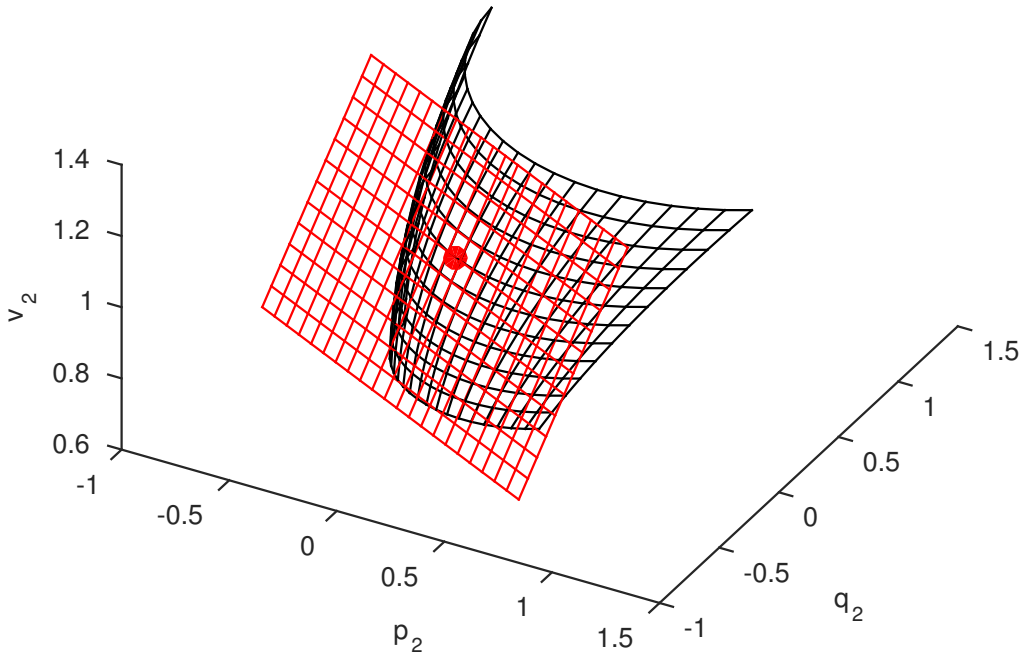
- line flow and squared voltages (DistFlow)
- rank-1 matrices of $\mathbb{C}^{n \times n}$
- ...

Each of these choices have advantages for a specific purpose:

- PF approximations
- relaxations
- numerical algorithms
- ...

Daniel K. Molzahn and Ian A. Hiskens
A Survey of Relaxations and Approximations of the Power Flow Equations
Foundations and Trends® in Electric Energy Systems, 4(1-2), 2019.

Power flow manifold approximation



Best linear approximant

$$A_{x^*}(x - x^*) = 0$$

$$A_{x^*} := \nabla F(x)|_{x=x^*}$$

Tangent plane at a nominal power flow solution $x^* \in \mathcal{M}$

$$\nabla F = \left[\left(\lfloor \mathrm{diag}(Y\vec{v}) \rceil + \lfloor \mathrm{diag}(\vec{v}) \rceil \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix} \right] \lfloor Y \rceil \right) R(v,\theta) \quad -I \right]$$

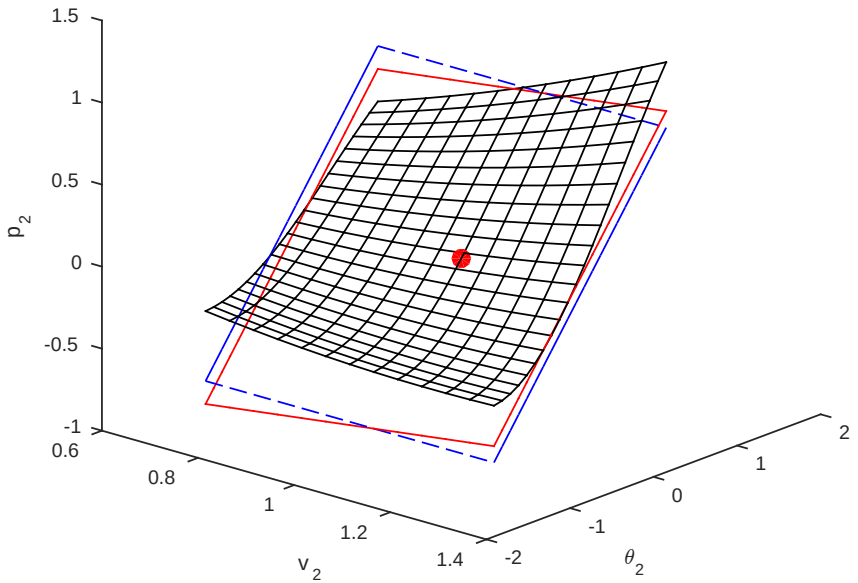
where

$$\lfloor A \rceil = \begin{bmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{bmatrix} \qquad R(v,\theta) = \begin{bmatrix} \mathrm{diag}(\cos\theta) & -\mathrm{diag}(v)\,\mathrm{diag}(\sin\theta) \\ \mathrm{diag}(\sin\theta) & \mathrm{diag}(v)\,\mathrm{diag}(\cos\theta) \end{bmatrix}$$

S. Bolognani and F. Dörfler
Fast power system analysis via implicit linearization of the power flow manifold
53rd Annual Allerton Conference on Communication, Control, and Computing, 2015.

PFM approximation

$$\nabla F = \left[\left(\lfloor \mathrm{diag}(Y\vec{v}) \rceil + \lfloor \mathrm{diag}(\vec{v}) \rceil \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \lfloor Y \rceil \right) R(v,\theta) \quad -I \right]$$



Standard models

By selecting a specific **linearization point** and via additional **modeling assumptions** one recovers

- linear coupled power flow
- DC power flow
- ...

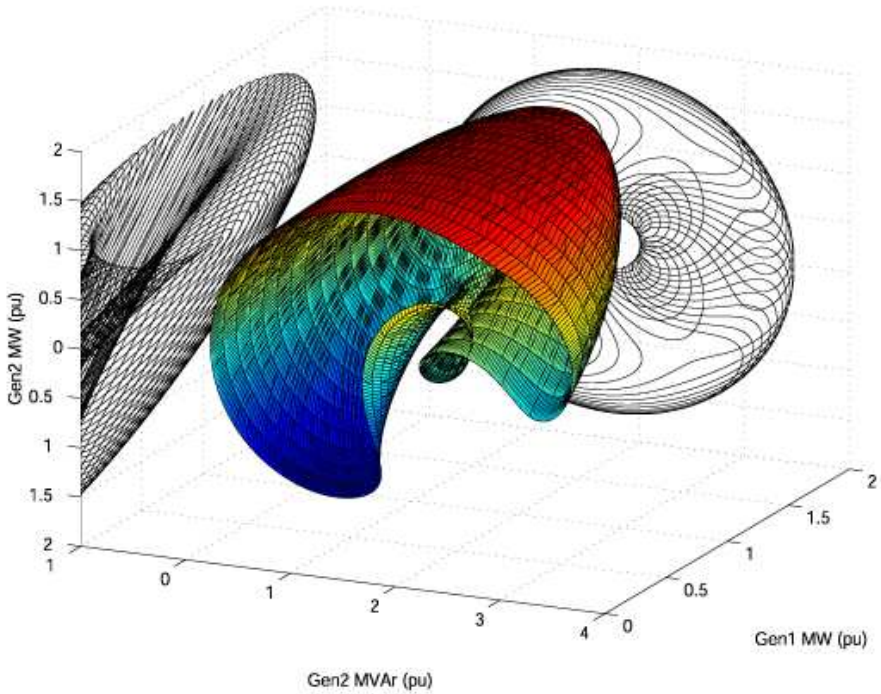
- **flat voltage profile** $\vec{v} = \mathbb{1}$

$$\nabla F = \begin{bmatrix} G & -B & -I & 0 \\ -B & -G & 0 & -I \end{bmatrix} \quad \text{that is} \quad \begin{bmatrix} G & -B \\ -B & -G \end{bmatrix} \begin{bmatrix} v \\ \theta \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

- **flat voltage profile** $\vec{v} = \mathbb{1}$ and **inductive lines** $Y = jB$

$$\rightarrow \text{DC power flow model} \quad \begin{cases} p = -B\theta \\ q = -Bv \end{cases}$$

PFM: take-home concepts



→ Hiskens and Davy, 2001

- Steady states of a power system form a **manifold of dimension** $2n$
- The PFM can be **embedded** in the v, θ, p, q space (with a **global chart**)
- An analytic expression for the **tangent plane** to the PFM exists
- The tangent plane can be used as a **linear approximant** of the PF eqs
- The analytic expression for the tangent plane is **numerically tractable** (sparse, structure preserving)