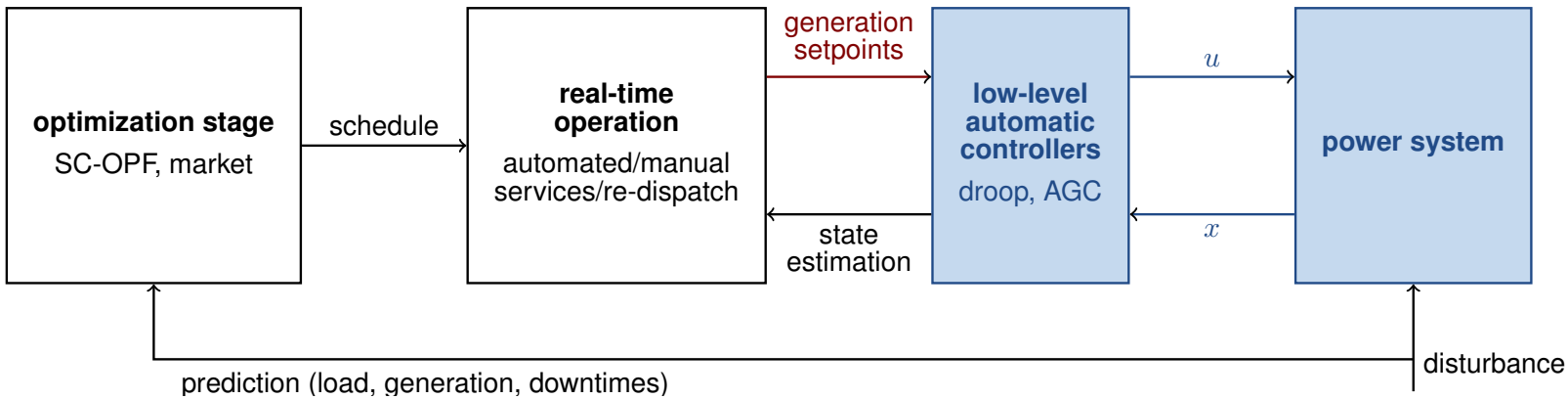


Real-time operation

Saverio Bolognani, ETH Zurich

EECI School M07, May 6-10, 2024, Barcelona

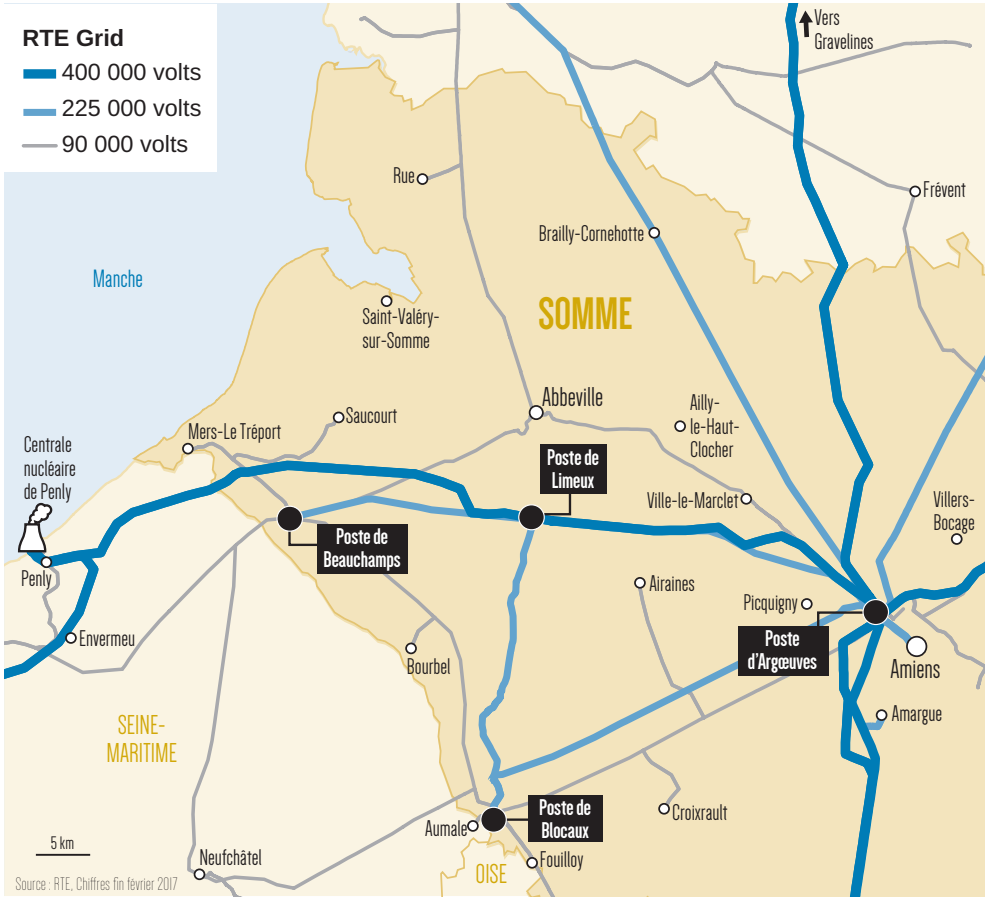
Real-time operation

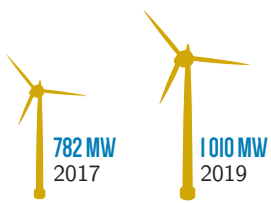


Power system dynamics are considered at **steady state**

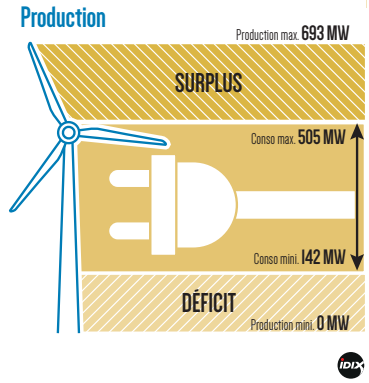
- electromechanical dynamics of the generators
  $\rightarrow$ PQ buses w/o voltage regulation, PV buses w/ voltage regulation
- loads
  $\rightarrow$ typically PQ buses, but possibly nonlinear static $p(v), q(v)$
- line dynamics
- low-level controllers
- frequency regulators (power balance)

Example: transmission grid congestion







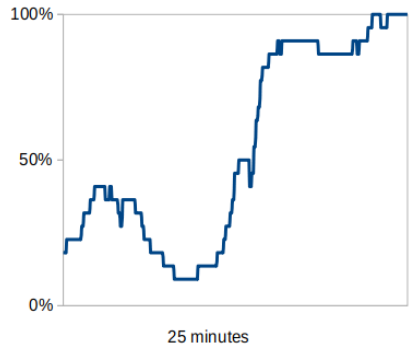


UNICORN

Rte

Schweizerische Eidgenossenschaft
Confédération suisse
Confederazione Svizzera
Confederaziun svizra

Swiss Federal Office of Energy SFOE



Rapid changes in generation/demand cause violation of

- line limits (depending on the severity, need to be resolved immediately or within minutes)
- voltage limits

Available actuation (=set-points)

- **Active power curtailment**
  - modulation vs full range curtailment (depending on the technology)
  - ramp up/down limits (inverters: $0s$, wind: $20s$ in emergency, $60s$ otherwise)
- **AVR (Automatic Voltage Regulators) set-points**
  - example: in France, remotely adjusted every $10s$
- **Active power injection from storage**
  - Minimal delay, high flexibility
- **Reactive power injection**
  - any inverter can participate (generators, batteries, loads)
  - hard reactive power limits depending on the inverter type and controller
  - causes losses in distribution grids
- Tap changers at the substation transformers
- Grid topology (connect/disconnect lines)
- Shunt admittances (connect/disconnect capacitor banks)
- FACTS (for example HVDC links)

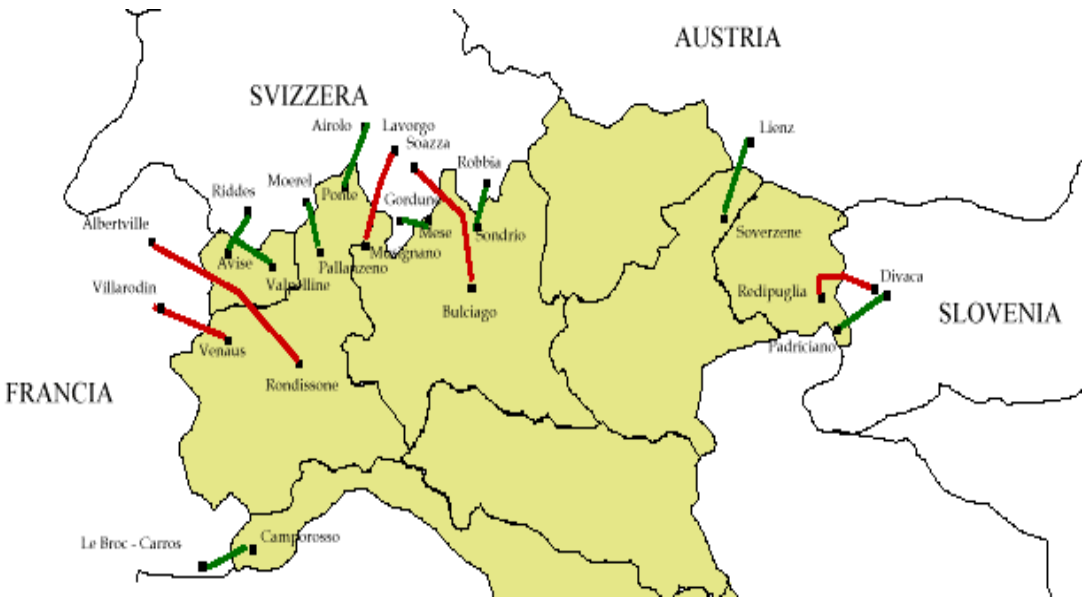
How to use these actuation signals (set-points) to solve congestion?

How is it done today?

- Many of these set-points are not controlled in real-time (example, AVR voltage set-points)
- Each class of actuators is controlled separately
- Most control strategies are
  - used only in emergencies
  - based on heuristics and operator knowledge / rulebooks based on simulations
  - mediated by human operators

How is it done today?

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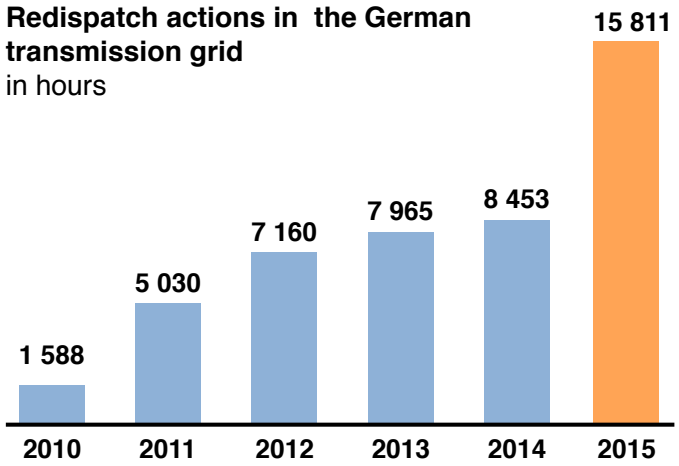


→ Overview of the events and causes of the 2003 Italian blackout

03:01	Trip of the 380 kV line Mettlen Lavorgo (CH). Attempts to reclose the line automatically until 3.03:50. Manual re-closure fails at 3:03:50.
03:02-03:08	Attempts to reclose the Mettlen-Lavorgo line. Information exchanges between ETRANS and ATEL and EGL dispatchers.
03:10	ETRANS, by phone, requests a reduction of 300 MW in Italian imports to scheduled values.
03:18-03:22	Exchange of information ETRANS - ATEL –EGL; changes in topology of the Swiss system.
03:21	Italian imports are reduced to 6400 MW
03:25	Trip of the Sils-Soazza 380 kV line (CH)
03:25	Trip of the Airolo Mettlen 220 kV line (CH)

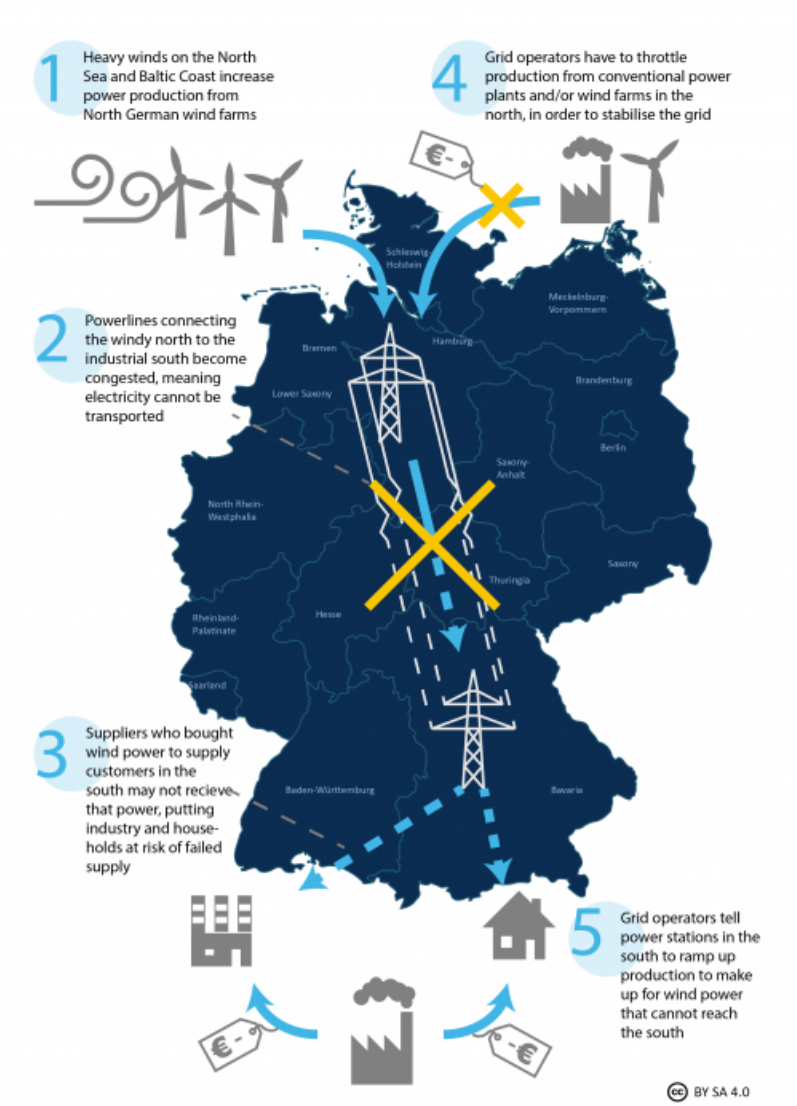
→ 1.2 billion EUR

Not only in emergencies



Ideally:

- Coordinated use of the available actuation
- Coordinated update of set-points
- A unified approach for multiple ancillary services



CLEW, 2016

$$$

US

Curtailment, congestion costs rise as transmission upgrades lag renewable growth

Despite increasing momentum for transmission line upgrades, questions remain whether these projects can match the pace of intermittent renewable generation coming online on the heels of the Inflation Reduction Act of 2022. Over 1,000 GW of wind and solar capacity are in interconnection queues across the seven ISOs in the US. Just a quarter of this capacity reaching commissioning would double the current operating fleet of these technologies.

Germany

13 Dec 2022, 13:41 Benjamin Wehrmann

Germany curtails four percent of renewable power production due to grid bottlenecks

29 Jan 2024, 13:44 Benjamin Wehrmann

Lack of grid capacity lets output of Germany's North Sea offshore wind turbines drop in 2023

Spain

GRID MANAGEMENT CHALLENGES COSTING SPANISH ENERGY CONSUMERS

April 12, 2023

Spain is undoubtedly a renewable energy superpower. Favourable weather conditions and ambitious government targets have enabled rapid buildout over the past decade. Renewables accounted for 51% of total power generation in 2022, compared to 46% in Germany and 38% in the UK, and installed capacity rose by 9% year on year. Aurora Energy Research forecasts that renewables will generate 80% of total power output in 2030, surpassing the government's target of 74%.

The national electricity transmission system has not expanded in line with renewable power generation—a rapidly rising number of generation assets are connecting to a disproportionate number of transmission assets, resulting in increasingly frequent periods where, in certain areas, electricity generation exceeds the capacity of local transmission assets and cannot be delivered to consumers. **These constraints are costing renewable generators and electricity consumers and are slowing progress towards Net Zero, new analysis by Aurora finds.**

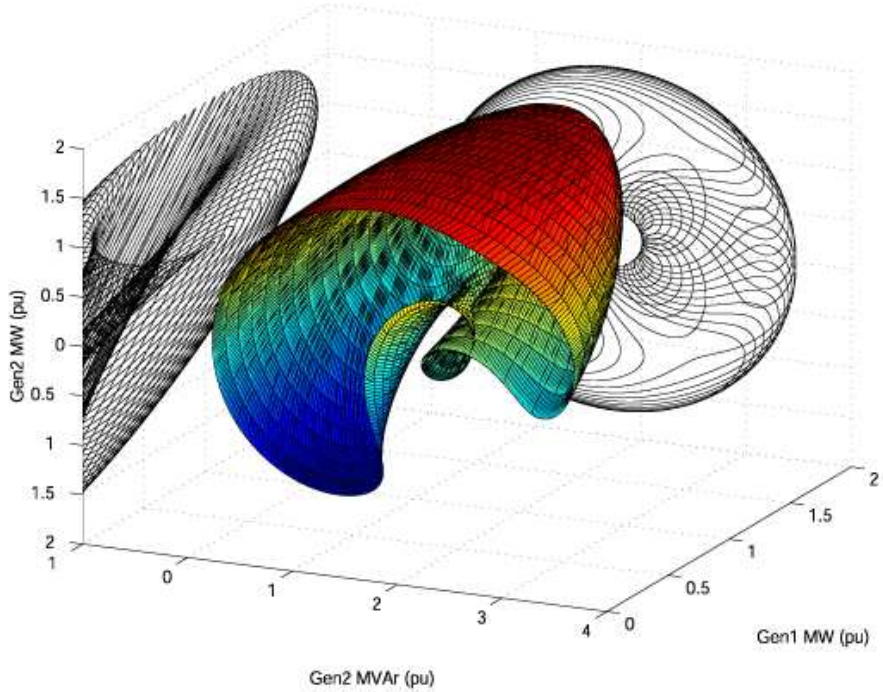
Outline

- Optimal power flow
- Real-time OPF
- Projected dynamical systems
- Gradient flow on the PFM

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Power Flow Manifold - recap



→ Hiskens and Davy, 2001

- Steady states of a power system form a **manifold of dimension** $2n$
- The PFM can be **embedded** in the v, θ, p, q space (with a **global chart**)
- An analytic expression for the **tangent plane** to the PFM exists
- The tangent plane can be used as a **linear approximant** of the PF eqs
- The analytic expression for the tangent plane is **numerically tractable**

Optimal set-points

Now that we have defined the set of all **valid set-points** in the $v - \theta - p - q$ space (PFM), we can introduce a concept of **optimality** on this set: **cost** + **constraints**

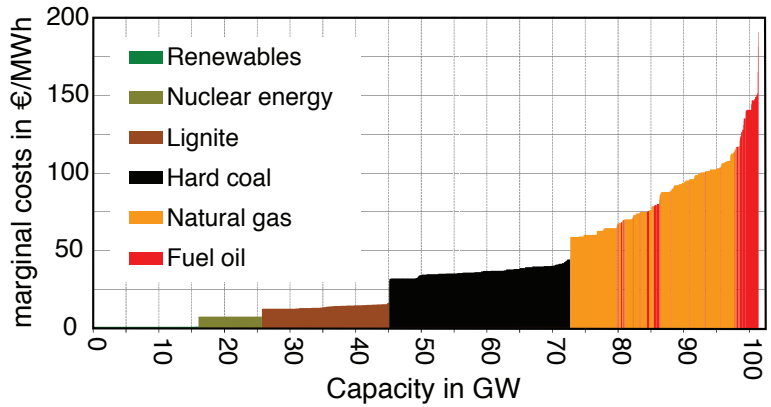
Optimal set-points

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Cost function (usually linear or quadratic)

$$\sum_k \text{cost}_k(p_k)$$

- scheduled generation ($\rightarrow$)
- reserves (spot price: ~ 50 EUR/MWh)
- curtailment (~ 90 EUR/MWh)
- load shedding (~ 23 kEUR/MWh)



Constraints

Constraints are due to

- Physical saturation of devices
- Saturation of local controllers
- Hard limits in the specifications

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Simple examples

- power generation limits
  $p_i \in [\underline{p}_i, \overline{p}_i]$
- reactive power limits
  $q_i \in [\underline{q}_i, \overline{q}_i]$
- voltage limits
  $v_i \in [\underline{v}_i, \overline{v}_i]$

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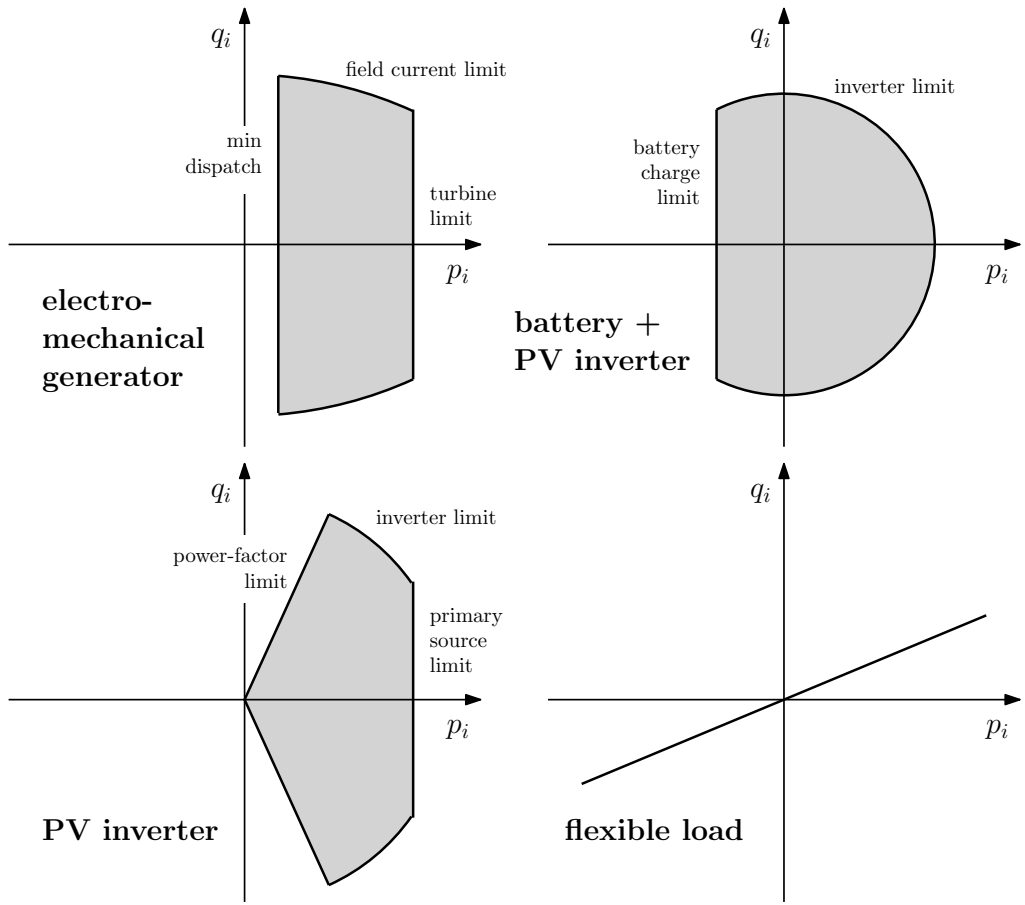
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In some cases, coupled constraints:

- device PQ regions

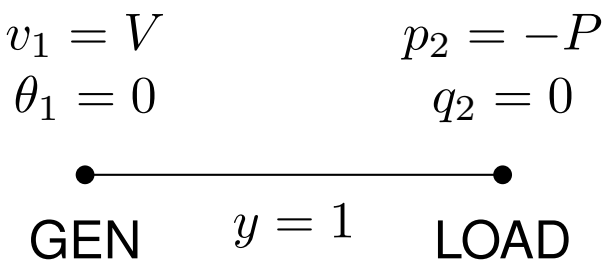


- line current limits $|i_{hk}| \leq \overline{i}_{hk}$

  typical proxy: θ_{hk}. Good enough?

Feasible region of the PFM

Example

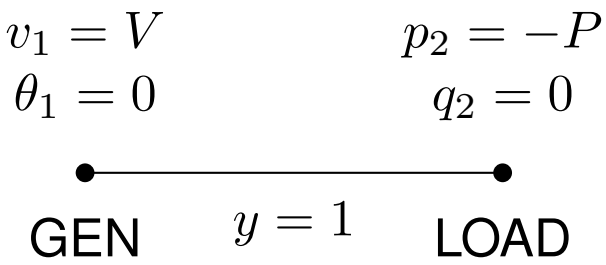


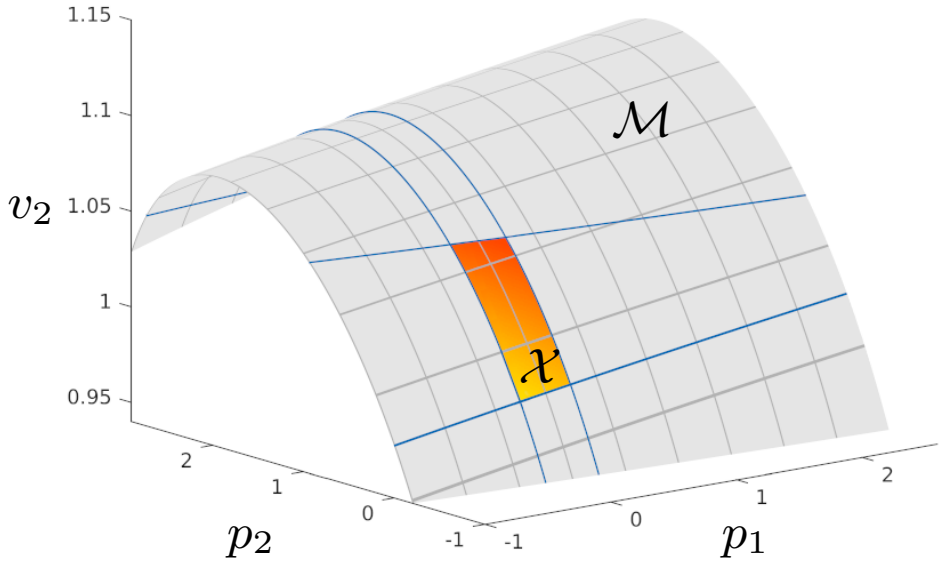
Set $\mathcal{K}$ defined by

- power generation limits
  $p_1 \in [-0.3, 0]$
- voltage limits
  $v_i \in [0.95, 1.05]$

Feasible region of the PFM

Example



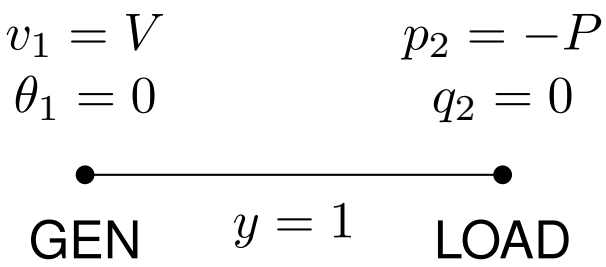


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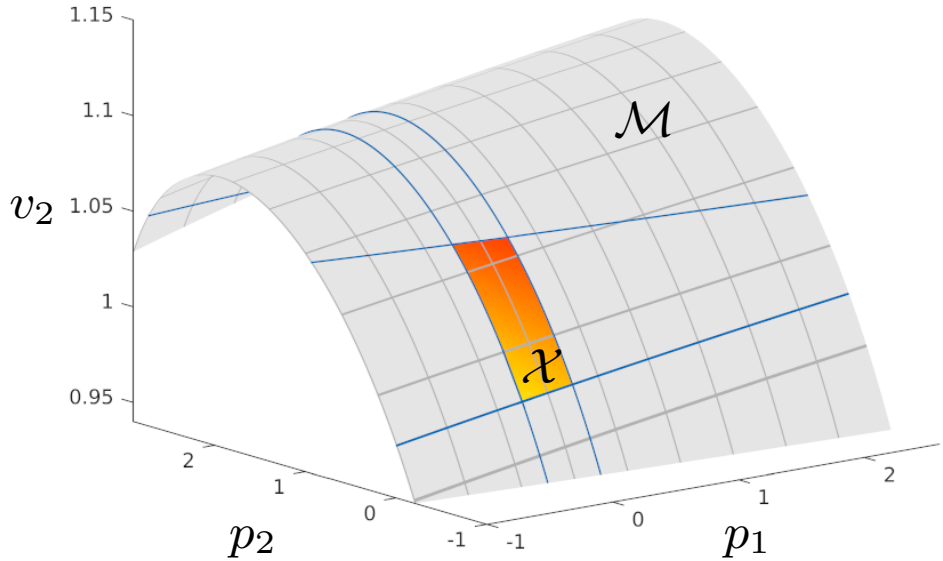
Feasible region of the PFM

Example



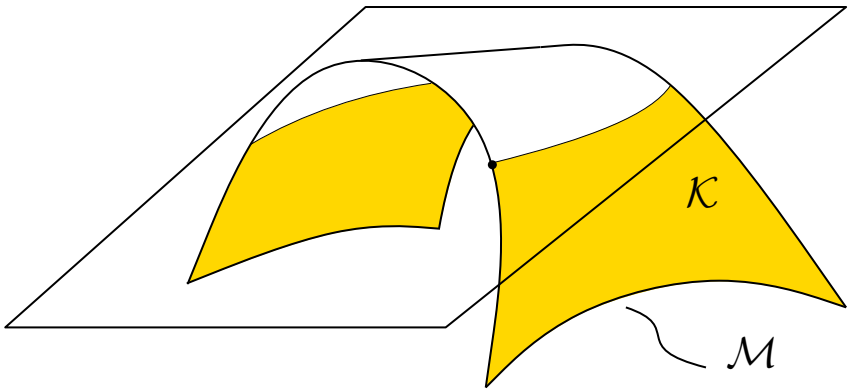
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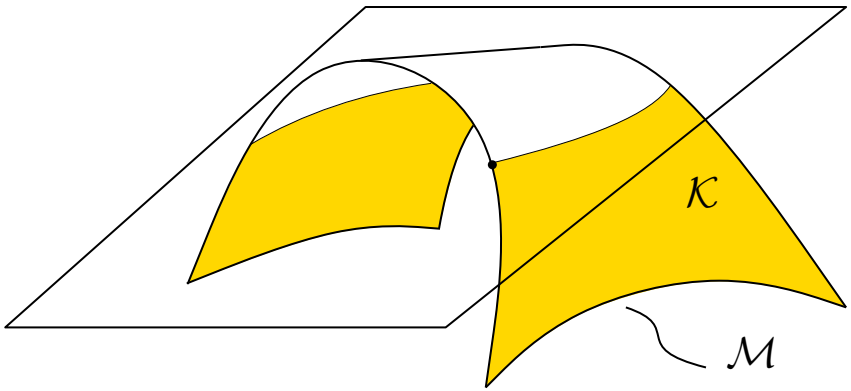
The region $\mathcal{X} = \mathcal{K} \cap \mathcal{M}$ of feasible set-points is a subset of the PFM $\mathcal{M}$ (non-convex subset of the ambient space).

Disconnected feasible regions



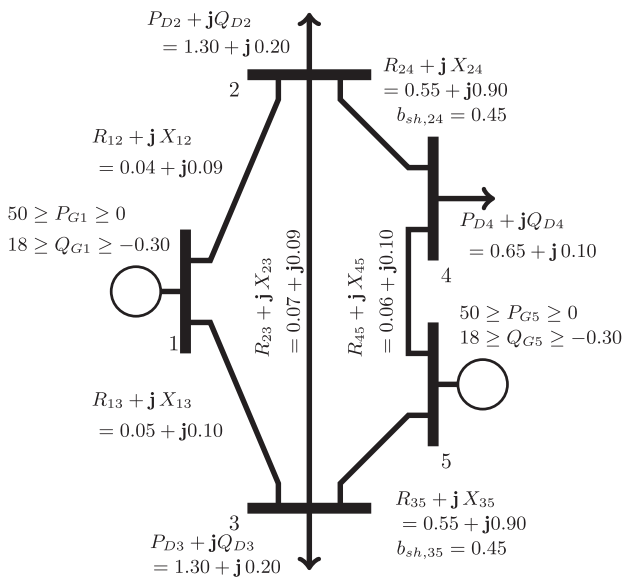
The intersection of the operational constraints $\mathcal{K}$ with the PFM can yield disconnected feasible regions

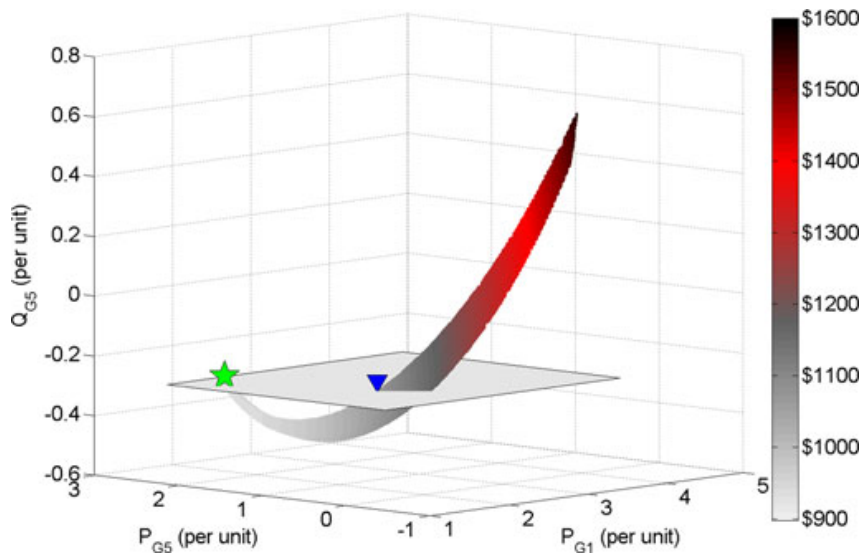
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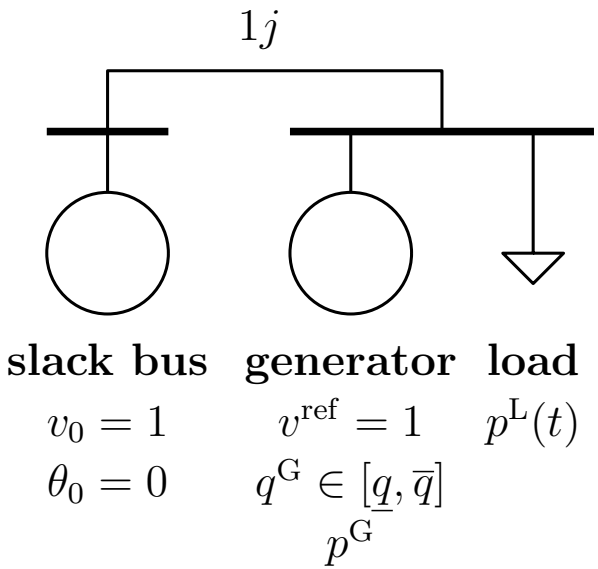
Daniel K. Molzahn
Computing the Feasible Spaces of Optimal Power Flow Problems
IEEE Transactions on Power Systems, 32(6), 2017.





Non-smooth feasible regions

Example: PV bus (generator) with limited reactive power capability



Union of 3 feasible regions

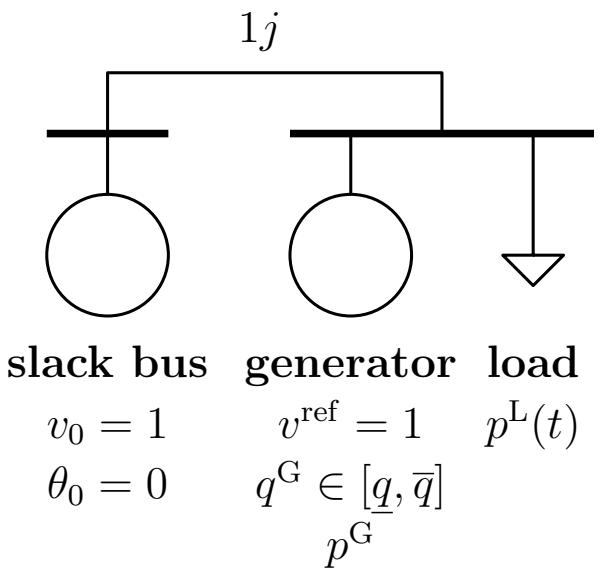
$$\mathcal{X}_1 = \mathcal{M} \cap \left\{ x \mid v_G = 1,\ q_G^{\min} \leq q_G \leq q_G^{\max} \right\}$$

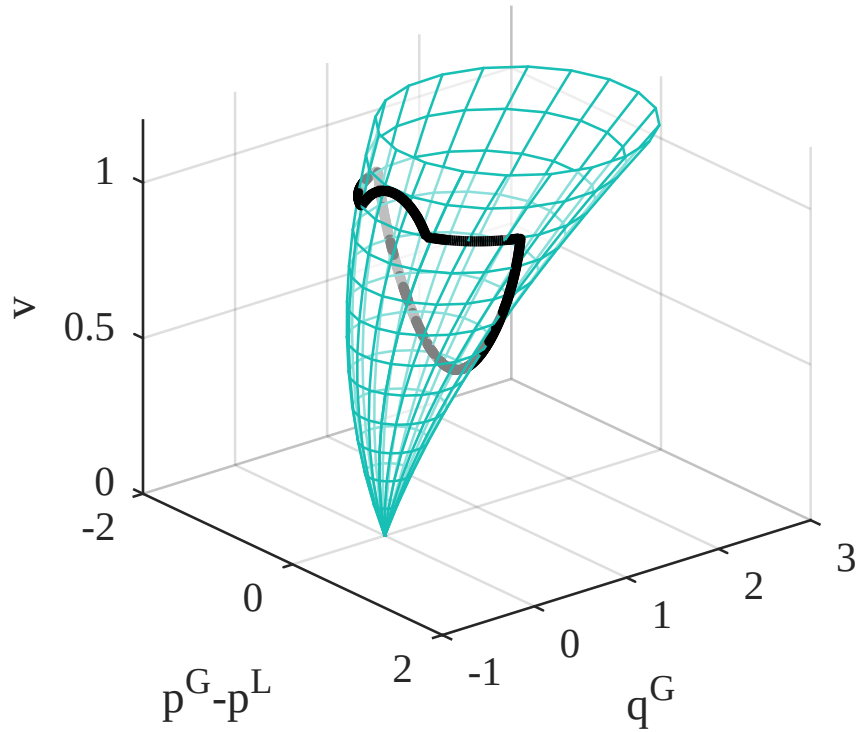
$$\mathcal{X}_2 = \mathcal{M} \cap \left\{ x \mid v_G \geq 1,\ q_G = q_G^{\min} \right\}$$

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Non-smooth feasible regions

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Optimal Power Flow

Traditional OPF formulation

- Minimize cost of generation
- Satisfy AC power flow eqs ($\mathcal{M}$)
- Respect generation capacity
- No over-/under-voltage
- No line congestion

$$
\begin{aligned}
\text{minimize} \quad & \sum\nolimits_{k \in \mathcal{N}} \text{cost}_k(p_k^G) \\
\text{subject to} \quad & p^G + jq^G = p^L + jq^L + \text{diag}(v) Y' v' \\
& \underline{p}_k \leq p_k^G \leq \overline{p}_k, \;\; \underline{q}_k \leq q_k^G \leq \overline{q}_k \\
& \underline{v}_k \leq v_k \leq \overline{v}_k \\
& |i_{hk}| \leq \overline{i}_{jl}
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$$

OPF

$$
\begin{aligned}
\text{minimize} \quad & \phi(x) \\
\text{subject to} \quad & x \in \mathcal{X} = \mathcal{M} \cap \mathcal{K}
\end{aligned}
$$

$\phi : \mathbb{R}^{4n} \to \mathbb{R}$	objective function
$\mathcal{M} \subset \mathbb{R}^{4n}$	manifold defined by AC power flow equations
$\mathcal{K} \subset \mathbb{R}^{4n}$	operational constraints

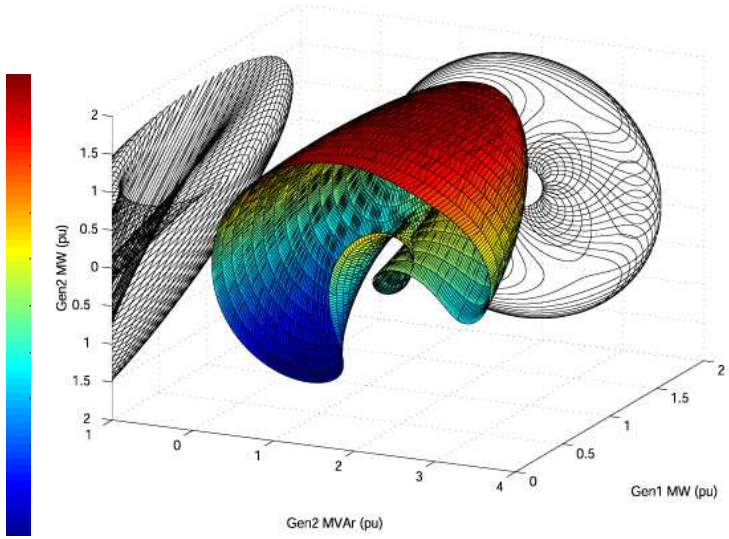
(embedding in $\mathbb{R}^{4n}$ is actually not necessary)

Optimization perspective

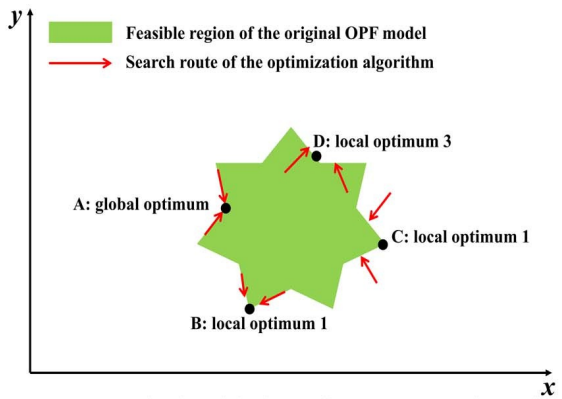
Non-convex program

$$\begin{aligned} \text{minimize} \quad & \phi(x) \\ \text{subject to} \quad & x \in \mathcal{X} = \mathcal{M} \cap \mathcal{K} \end{aligned}$$

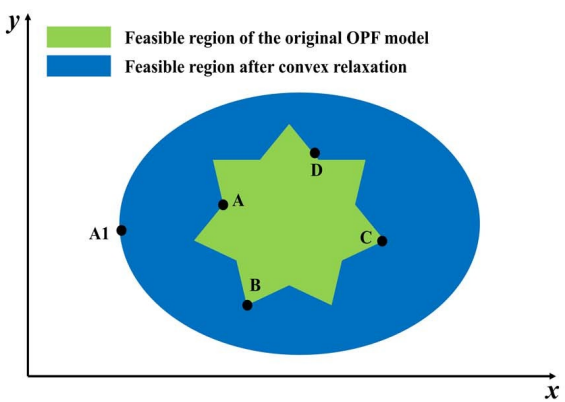
Its complexity comes from the AC constraints ($\mathcal{M}$). Everything else (ϕ, $\mathcal{K}$) is mostly trivial.



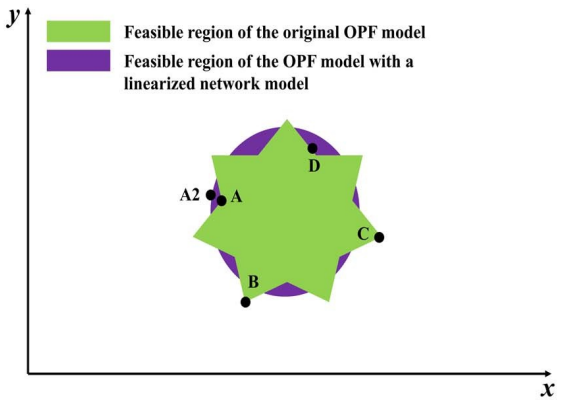
Exact AC power flow eqs



Convex relaxation



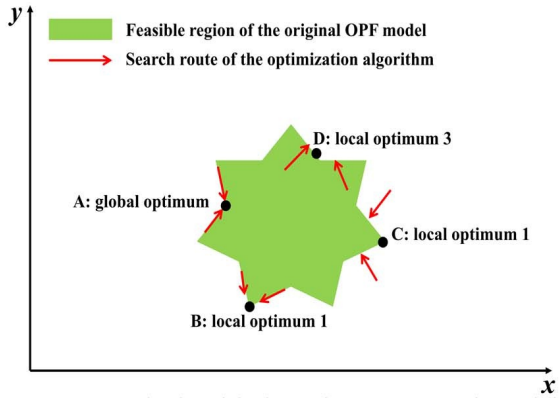
Approximation/linearization



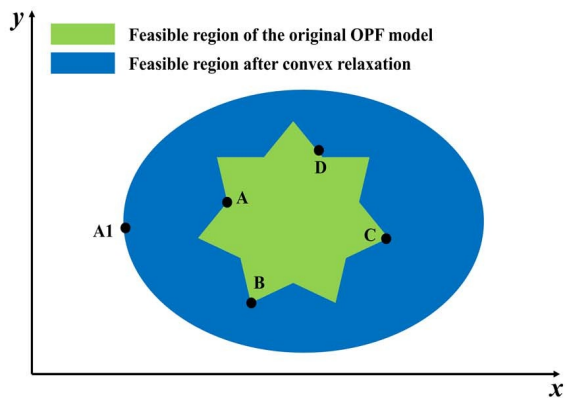
→ Fundamental Review of the OPF Problem: Challenges, Solutions, and State-of-the-Art Algorithms (2017)

OPF approaches

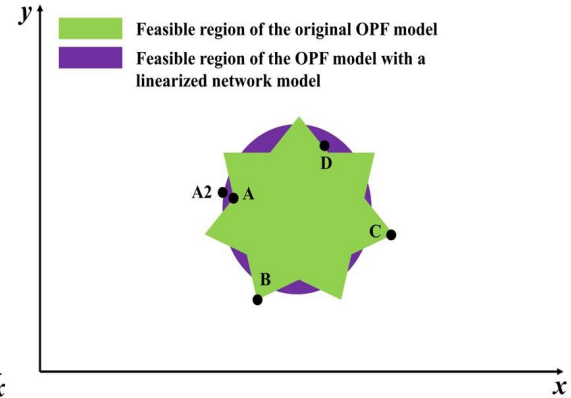
Exact AC power flow eqs



Convex relaxation



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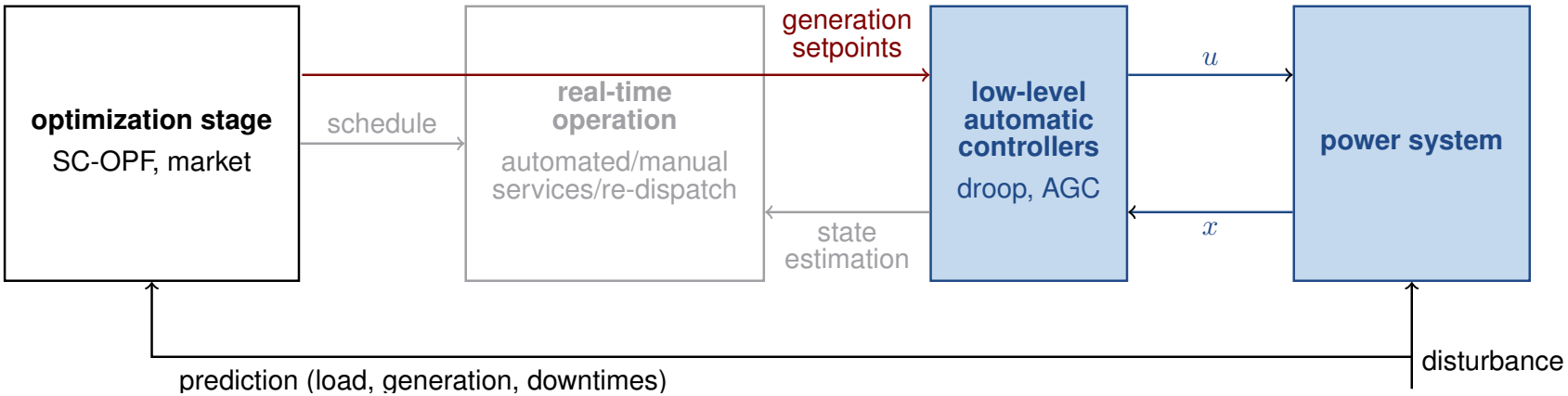
Properties	OPF methods with strict AC network model	OPF methods based on convex relaxation	OPF methods with linearized network models
Computational difficulty	NP hard problem	Terminate in a polynomial time (for most cases, the rank of computational difficulty is SOCP < QP < SDP)	Terminate in a polynomial time
Convergence	Not guaranteed	Guaranteed	Guaranteed
Solution quality	Obtained solution strictly subject to power flow equations and operational limits; obtained solution may be a local optimum	When relaxations are inexact, the physical meaning of the obtained solutions is unclear	Solutions are not strictly subject to power flow equations, but are close to be AC feasible
Industry preference	Currently used for the optimization of reactive power, such as in the tertiary voltage control	No evidence of industrial applications	Currently used in the clearing engine of power markets and power system planning, among others

Daniel K. Molzahn and Ian A. Hiskens
A Survey of Relaxations and Approximations of the Power Flow Equations
Foundations and Trends® in Electric Energy Systems, 4(1-2), 2019.

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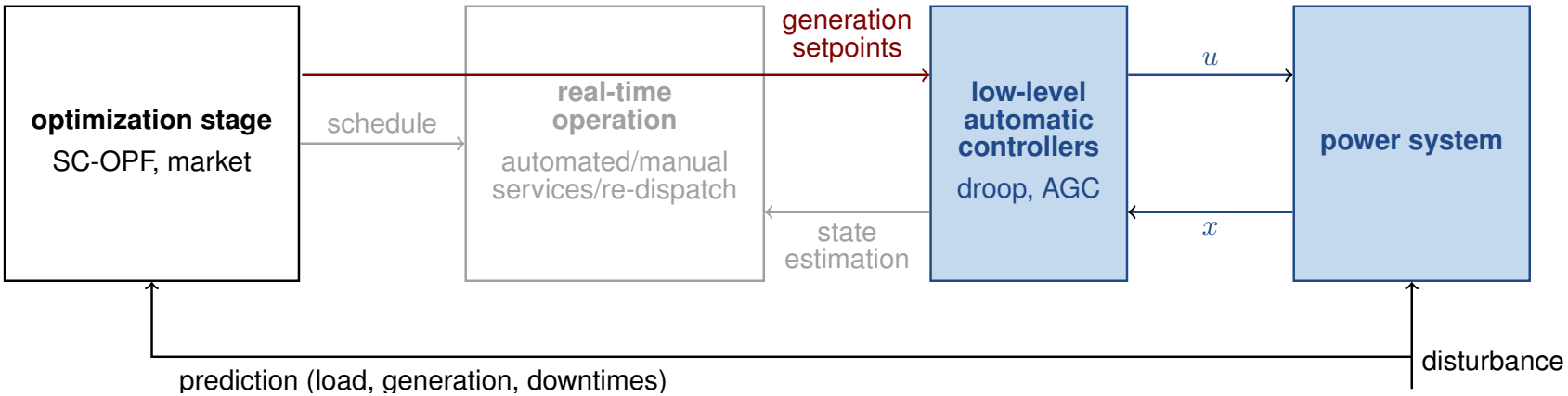
“Repeated” OPF



Instead of preparing a **schedule**, compute the optimal set-points based on

- the best available estimate of the **exogenous inputs**
- a model of the power system

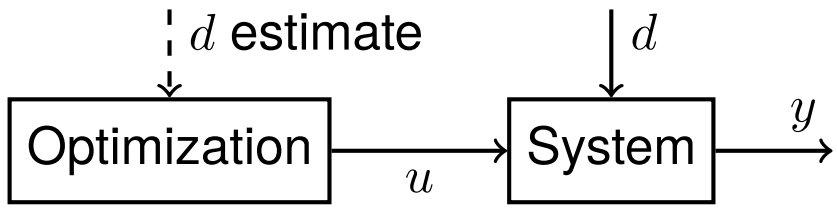
“Repeated” OPF



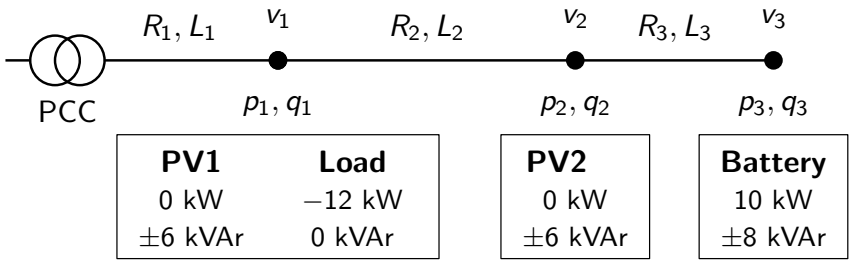
Instead of preparing a **schedule**, compute the optimal set-points based on

- the best available estimate of the **exogenous inputs**
- a model of the power system

In control theory terms:
feedforward



A simple experiment

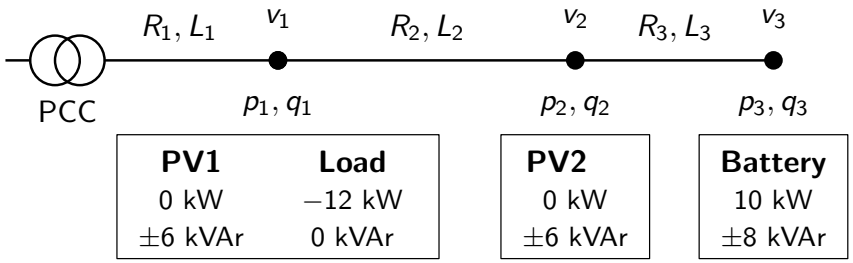


TEAMVAR

ERIGrid Research Infrastructure
EU Horizon 2020

→ Experimental Validation of Feedback Optimization in Power Distribution Grids (2020)
in collaboration with DTU, Denmark

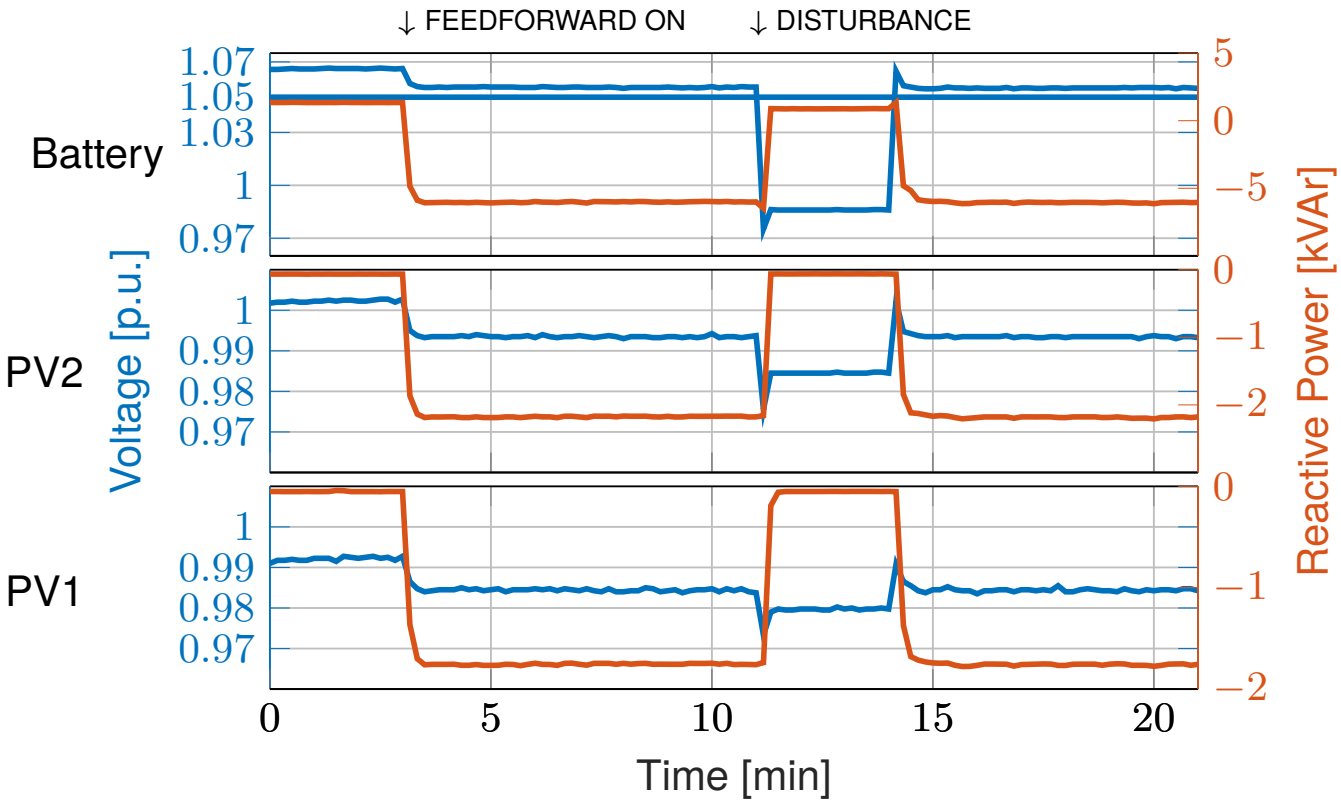
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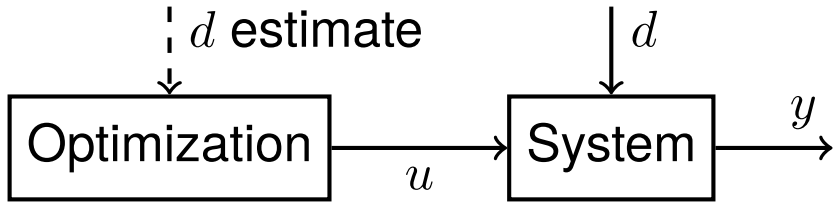
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Drawbacks of feedforward



- Sensitivity to **model mismatch** and **disturbances**
  - AC-unfeasible solution ($x \notin \mathcal{M}$)
  - constraint violations ($s \notin \mathcal{K}$)
  - suboptimality
- **No trajectory** to the new optimal point
  - ill-behaved (possibly discontinuous) dependence of the solution on the problem parameters
  - unfeasible actuation (remember: disconnected $\mathcal{X}$)
- **Computationally challenging**
  - AC feasibility $\Rightarrow$ NP-hard nonconvex optimization

Trajectories on the PFM

We are not using a lot of **useful structure** of the problem!

- the search space is a manifold
  $\rightarrow$ **locally**, it is a very tractable search space ($\sim \mathbb{R}^{2n}$)

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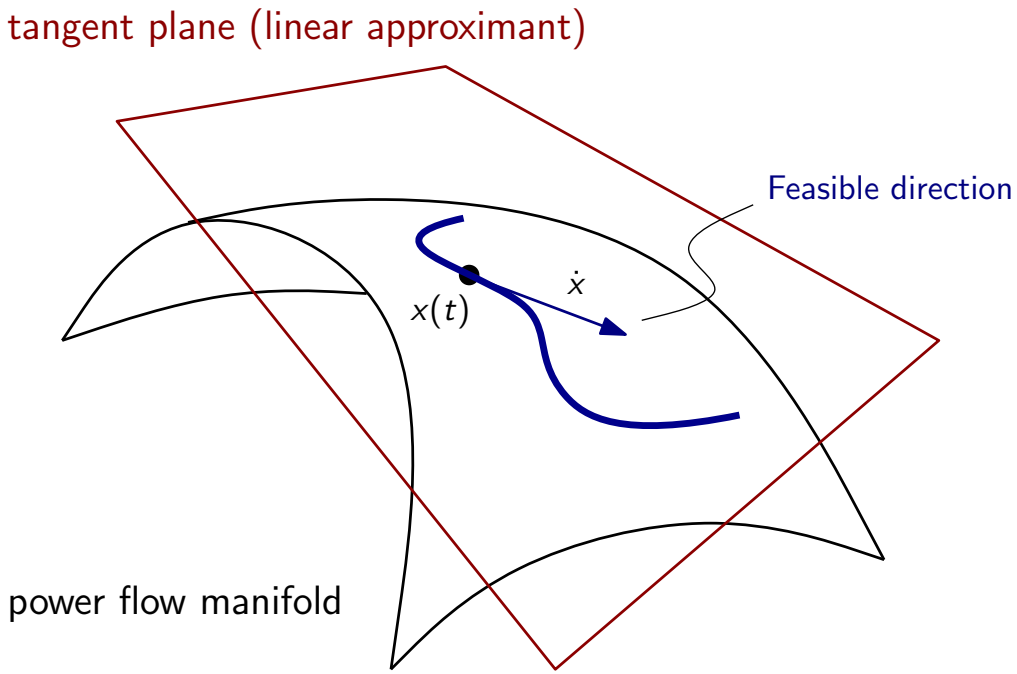
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  $\rightarrow$ **feasible directions** belong to the tangent plane (known in closed form)

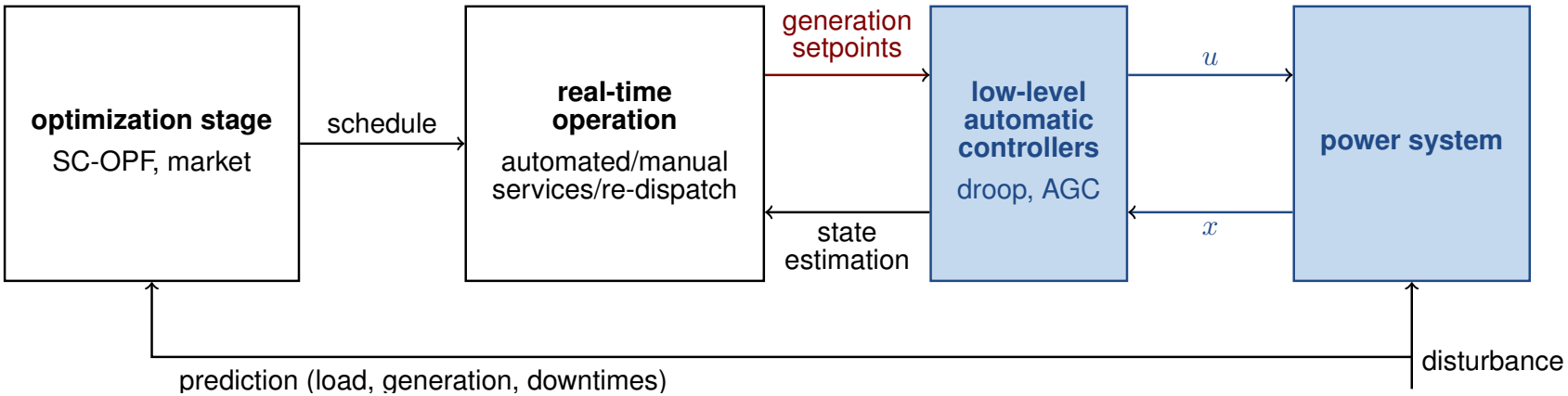
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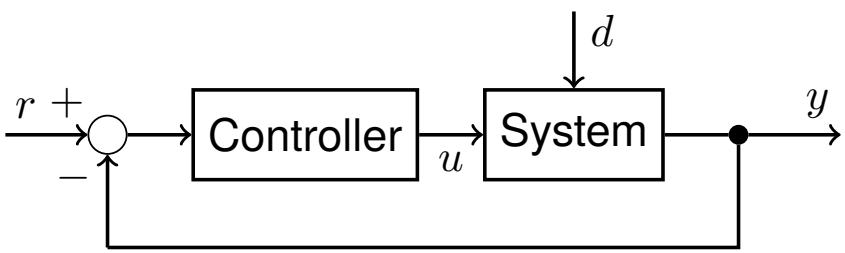
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A control interpretation of real-time OPF



In control theory terms:
feedback



- closed-loop trajectory belongs to $\mathcal{X}$ at all times
- numerically trivial (no PF solver!)
- **goal: closed-loop dynamics**

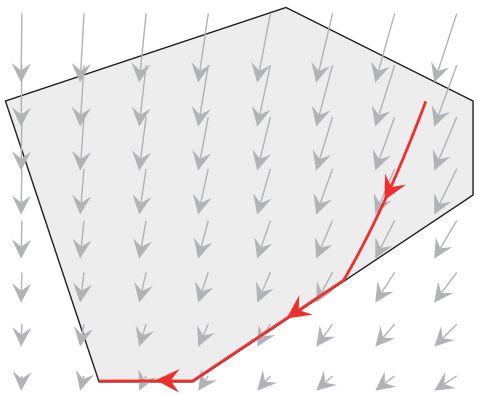
$$\dot{x} = f(x)$$

for which the OPF solution is an asymptotically stable equilibrium

Outline

- Optimal power flow
- Real-time OPF
- Projected dynamical systems
- Gradient flow on the PFM

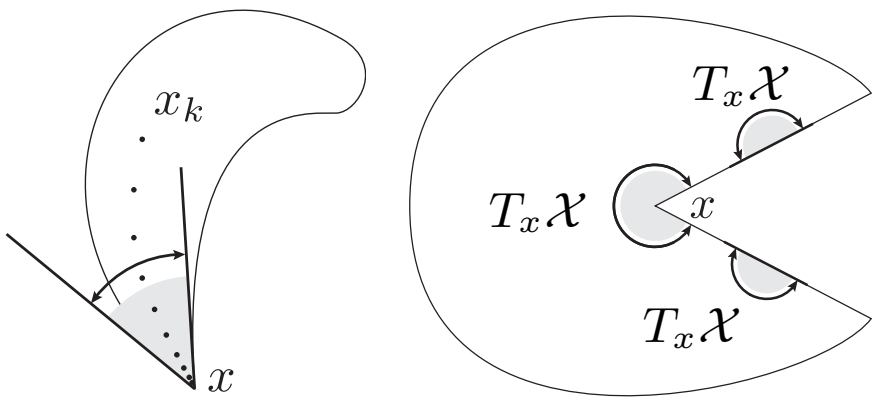
Projected dynamical systems



Intuitively...

- $\dot{x} = f(x)$ in the interior of $\mathcal{X}$
- $x(t)$ "slides" on the boundary of $\mathcal{X}$
- $x(t)$ never leaves $\mathcal{X}$

Tangent cone: $$T_x\mathcal{X} := \left\{ v \;\middle|\; \exists x_k \underset{\mathcal{X}}{\to} x, \delta_k \to 0^+ : \frac{x_k - x}{\delta_k} \to v \right\}$$



Clarke regularity

A set is Clark regular if its boundary is smooth, except for convex edges and corners.

Formally: $\liminf_{y \to x} T_y\mathcal{X} = T_x\mathcal{X}$

Is $\mathcal{X}$ Clarke-regular?

Consider a set

$$\mathcal{X} = \{x \mid h(x) = 0,\ g(x) \leq 0\}$$

where h **and** g **are continuously differentiable**. **If LICQ is satisfied**, i.e.,

$$\begin{bmatrix} \nabla h(x) \\ \nabla g_{\text{active}}(x) \end{bmatrix}$$ full rank (i.e., **active constraints are linearly independent**)

at all points $x \in \mathcal{X}$ then $\mathcal{X}$ **is Clarke-regular**. $\rightarrow$ **Examples**

Moreover

$$T_x\mathcal{X} = \{v \mid \nabla h(x)v = 0,\ \nabla g_{\text{active}}(x)v \leq 0\}$$

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Good news

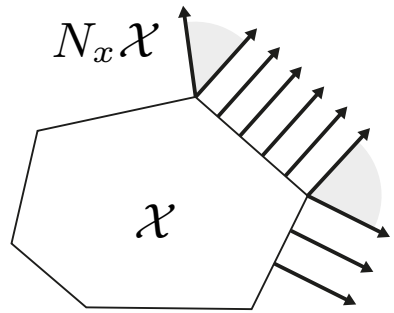
LICQ are **generically** satisfied in optimal power flow programs (i.e. with probability 1 when problem parameter – loads, shunts, ... – are randomly perturbed).

A. Hauswirth, S. Bolognani, G. Hug, F. Dörfler
Generic Existence of Unique Lagrange Multipliers in AC Optimal Power Flow
IEEE Control Systems Letters, 2(4):791-796, 2018. $\rightarrow$ preprint

Constrained optimization

For a Clarke-regular set, we define the **normal cone:**

$$N_x\mathcal{X} := \{\eta \,|\, \langle \eta, v\rangle \leq 0 \; \forall v \in T_x\mathcal{X}\}$$



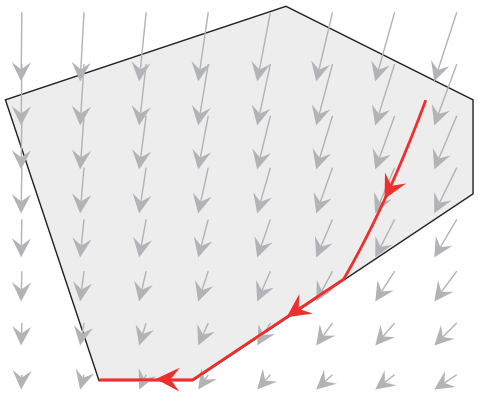
Normal cones and constrained optimization

Consider the problem $\min_{x\in\mathcal{X}} \; \phi(x)$ with $\mathcal{X}$ Clarke regular. Every local minimizer $x^\star$ satisfies

$$-\nabla\phi(x^\star) \in N_{x^\star}\mathcal{X}.$$

In the case $\mathcal{X} = \{x \mid h(x) = 0,\; g(x) \leq 0\}$, this is equivalent to the KKT conditions.

Projected dynamical systems



Initial value problem

$$\dot{x} \in \Pi_{\mathcal{X}}[f](x)\,, \qquad x(0) = x_0 \in \mathcal{X}$$

where

$$\Pi_{\mathcal{X}}[f](x) := \operatorname*{arg\,min}_{v \in T_x \mathcal{X}} \|v - f(x)\|$$

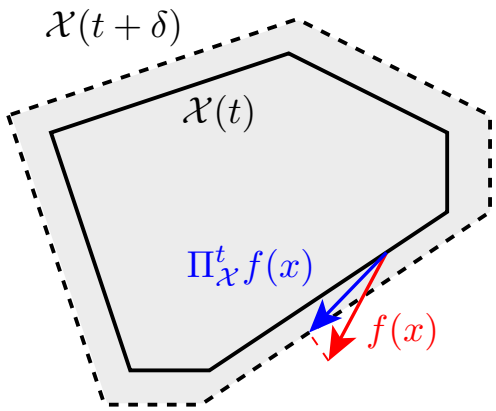
$\mathcal{X}$ Clarke-regular $\rightarrow$ $\Pi_{\mathcal{X}}[f](x)$ singleton $\rightarrow$ **ODE**

	domain $\mathcal{X}$	vector field f	metric g	manifold $\mathcal{M}$
existence of Krasovskii solutions	compact	C^0	C^0	C^1
existence of Carathéodory solutions, $\mathcal{S}_C = \mathcal{S}_K$	Clarke reg.	C^0	C^0	C^1
uniqueness of solutions	prox reg.	$C^{0,1}$	$C^{0,1}$	$C^{1,1}$

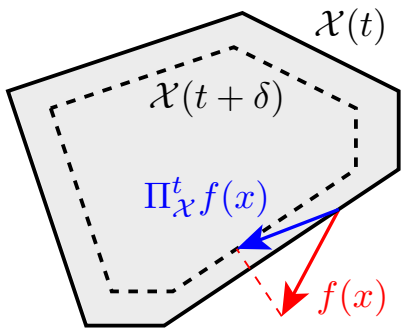
A. Hauswirth, S. Bolognani, F. Dörfler
Projected Dynamical Systems on Irregular, Non-Euclidean Domains for Nonlinear Optimization
arXiv:1809.04831 [math.OC]

Time-varying projected dynamical systems

expanding domain



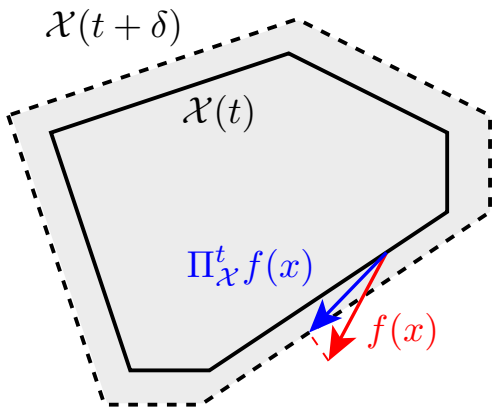
shrinking domain



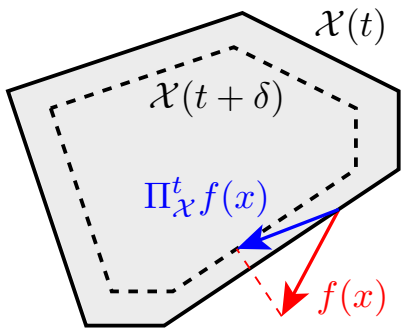
Can you think of some examples?

Time-varying projected dynamical systems

expanding domain



shrinking domain



Can you think of some examples?

New definition of **temporal tangent "cone"**

$$v \in T^t_x \mathcal{X} \Leftrightarrow \begin{cases} \exists x_k \to x, \delta_k \to 0^+ : \\ \frac{x_k - x}{\delta_k} \to v \quad \text{and} \quad x_k \in \mathcal{X}(t+\delta_k) \end{cases}$$

$\to$ not necessarily a cone (it can be empty)

Define **time-varying** projected vector field

$$\Pi_{\mathcal{X}}[f](x,t) := \underset{v \in T^t_x \mathcal{X}}{\arg\min} \|v - f(x)\|$$

$\to$ initial value problem

$$\dot{x} \in \Pi_{\mathcal{X}}[f](x)\,, \qquad x(0) = x_0 \in \mathcal{X}$$

Is it still a well-posed PDS?

Forward-Lipschitz continuity

Definition

The domain $\mathcal{X}(t)$ is **L-forward Lipschitz continuous** if for all t and $\delta \to 0^+$

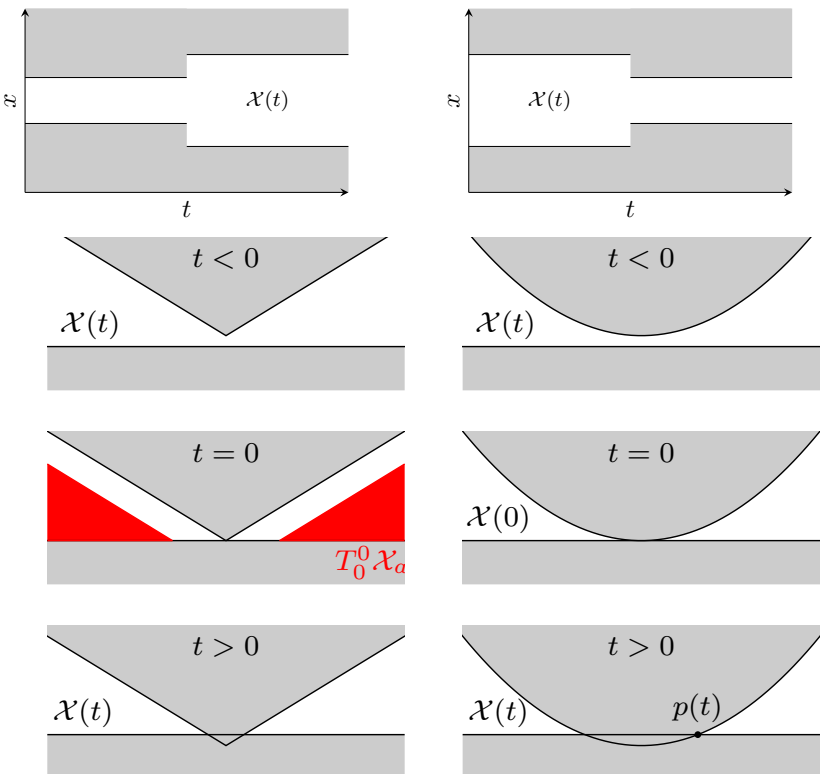
$$\mathcal{X}(t) \subseteq \mathcal{X}(t+\delta) + \delta L \mathbb{B}$$

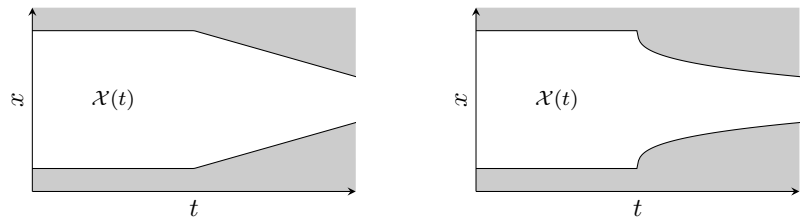
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- Which ones of these sets $\mathcal{X}$ are forward Lipschitz continuous?
- Is it enough to forbid discontinuous shrinkage?
  (e.g. line tripping, generator failure)
- Is it enough to limit the rate-of-change of individual constraints?
  (e.g. via ramp constraints)

Well-posednedd of time-varying PDS

Proposition

If the domain $\mathcal{X}(t)$ is **L-forward Lipschitz continuous**, then for every (t, x)

$$T_x^t \mathcal{X} \cap L\mathbb{B} \neq \emptyset \,.$$

Theorem

- Lipschitz continuous vector field $f(x)$
- **Lipschitz forward continuous** and closed feasible set $\mathcal{X}(t)$

$\rightarrow$ **Existence** of complete (Krasovskii) solution to the the initial value problem

$$\dot{x} \in \Pi_{\mathcal{X}}[f](x, t) \,, \quad x(0) \in \mathcal{X}(0)$$

What does it mean that a solution “does not exist”? Do we care in practice?

A. Hauswirth, I. Subotić, S. Bolognani, G. Hug, F. Dörfler
Time-varying PDS with Applications to Feedback Optimization of Power Systems
IEEE Conference on Decision and Control (CDC), 2018. $\rightarrow$ preprint

The projected dynamical system we need

Goal

$$\begin{aligned} \text{minimize} \quad & \phi(x) \\ \text{subject to} \quad & x \in \mathcal{X} \end{aligned}$$

If $\mathcal{X}$ is Clarke regular (which is generically true), then every local minimizer $x^\star$ satisfies

$$-\nabla\phi(x^\star) \in N_{x^\star}\mathcal{X}.$$

$$\dot{x} = \Pi_{\mathcal{X}}[f](x)\,, \qquad x(0) = x_0 \in \mathcal{X}$$

- $x(t)$ exists and stays in $\mathcal{X}$
- $\Pi_{\mathcal{X}}[f](x) = f - \eta, \quad \eta \in N_x\mathcal{X}$
- at rest when $\Pi_{\mathcal{X}}[f](x) = 0$ $\rightarrow f \in N_x\mathcal{X}$

What is a good candidate for f?

Outline

- Optimal power flow
- Real-time OPF
- Projected dynamical systems
- Gradient flow on the PFM

Review: unconstrained gradient flow

Theorem

If $\phi(x)$ has compact level sets, all trajectories of $\dot{x} = -\nabla\phi(x)$ are complete and converge to the set $\{x \mid \nabla\phi(x) = 0\}$.

Proof:

Review: unconstrained gradient flow

Theorem

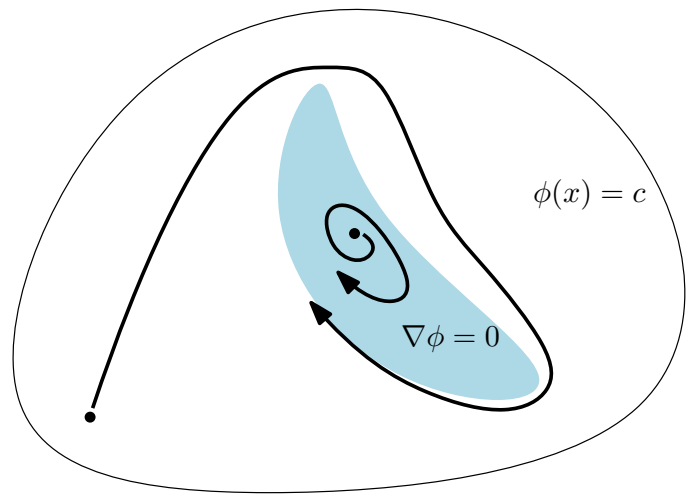
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Proof:

In general

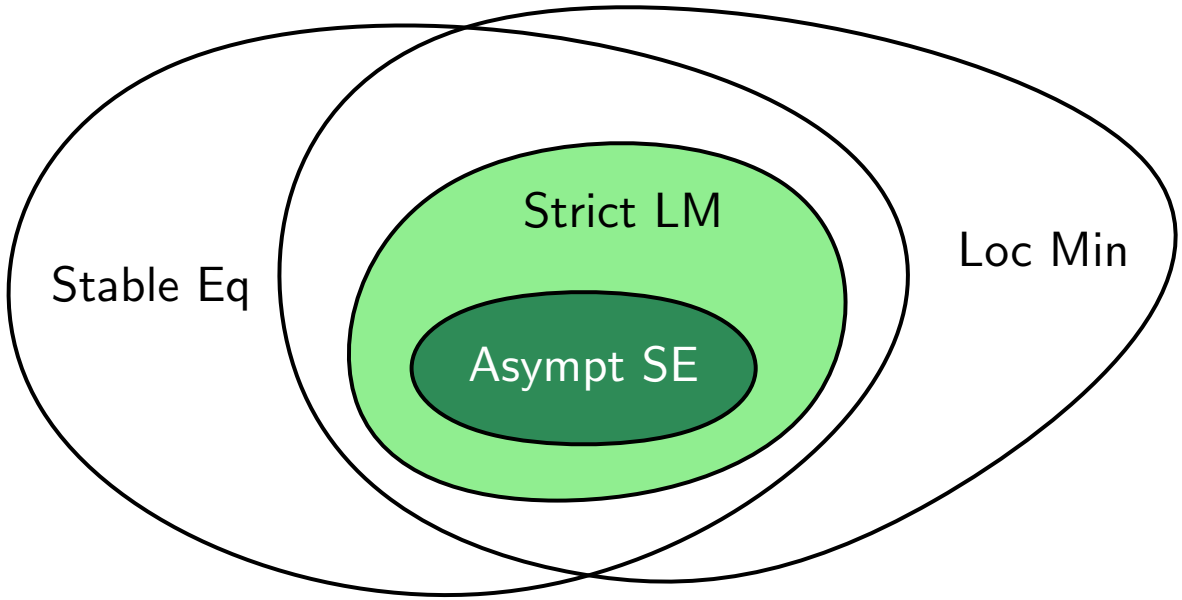
- trajectories are not guaranteed to converge to minimizers of $\phi(x)$
- minimizers of $\phi(x)$ are not guaranteed to be stable

Convex $\phi \to$ convergence to a global minimizer (trivial under **strict convexity**).



Critical points and stability

Without convexity it is still possible to identify minimizers based on their **stability properties** as dynamic equilibria.



Asymptotic stability of an equilibrium replaces the need for second-order optimality conditions (PD Hessian of $\phi(x)$)

P. A. Absil and K. Kurdyka
On the stable equilibrium points of gradient systems
Systems & Control Letters, 55(7), 2006.

Projected gradient

$$\dot{x} = \Pi_{\mathcal{X}}[-\nabla\phi](x)$$

- $\Pi_{\mathcal{X}}[-\nabla\phi](x) = -\nabla\phi - \eta, \quad \eta \in N_x\mathcal{X}$
- $\dot{x} = 0$ when $\Pi_{\mathcal{X}}[-\nabla\phi](x) = 0 \Leftrightarrow -\nabla\phi \in N_x\mathcal{X}$

Equilibria

Every critical point of $\min_{x\in\mathcal{X}} \phi(x)$ is a weak equilibrium, and every weak equilibrium is a critical point.

Lemma

$\phi(x)$ is non-increasing along trajectories of $\dot{x} = \Pi_{\mathcal{X}}[-\nabla\phi](x)$.

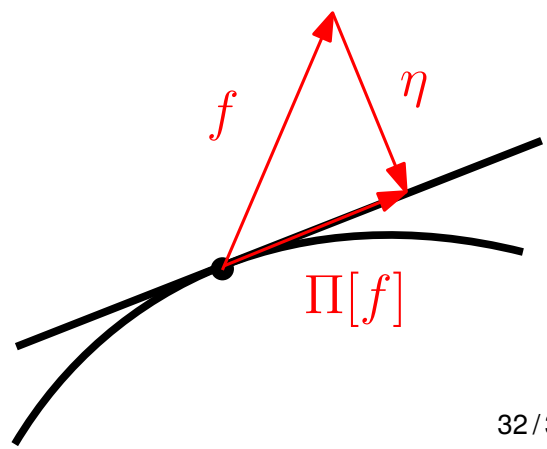
Convergence result

The projected dynamical system $\dot{x} = \Pi_{\mathcal{X}}[-\nabla\phi](x)^{\top}$ converges to the set of critical points of $\min_{x\in\mathcal{X}} \phi(x)$. Moreover

- asymptotically stable equilibria are strict minimizers
- strict minimizers are stable

Proof of the lemma (sketch)

Simplifying assumption: $\mathcal{X}$ Clarke-regular, unique solution. $\rightarrow$ Allerton 2016



Take-home

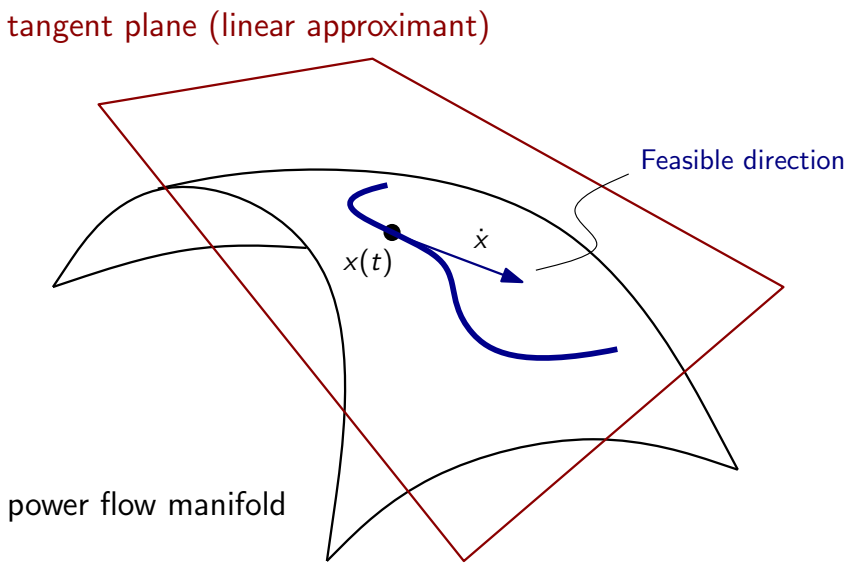
- We derives assumptions for the existence of **trajectories on the PFM**
  - Clarke regularity
  - Sufficient: Linear Independence Constraint Qualification (LICQ)
  - Time-varying: Forward-Lipschitz continuity

Take-home

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  - some interesting open questions

Take-home

- We derives assumptions for the existence of **trajectories on the PFM**
  - Clarke regularity
  - Sufficient: Linear Independence Constraint Qualification (LICQ)
  - Time-varying: Forward-Lipschitz continuity
- These conditions are **generically satisfied** in power systems
  - some interesting open questions
- Well-posedness of the PDS allow us to design **feedback optimization** strategies
  - example: projected gradient
  - classical (but powerful) tools: invariance principle



5-bus example, revisited

A lot of work, but we got the machinery that we need!

5-bus example, revisited

A lot of work, but we got the machinery that we need!

Same example as before:
disconnected $\mathcal{X}$

